Math circle

1 Pick five numbers from the set $\{1,2,3,4,5,6,7,8\}$. Then two of them must add up to 9.

Solution: We will apply pigeonhole principle.
We are looking for pairs of integers between 1 and 8 which have their sum equal 9.
The possibilities are

$$\boxed{1+8} \quad \boxed{2+7} \quad \boxed{3+6} \quad \boxed{4+5} \qquad (1)$$

These are the holes in the pigeonholes principle.
The pigeons are the five numbers we pick.
Out of these 5 numbers at least two should be in the same hole in (1)
Thus these two will have their sum equal 9 since they are in the same box.

Consider the following related problem from the book "Putnam and Beyond".

Problem 33 / page 12

The solution of this problem is similar and it will be a good excercise.

2. In a group of six people there will allways be 3 people that are mutual friends or mutual strangers
(We assume that if x is friend to y then y is friend to x and that if x and y are not friends they are strangers).

Solution: Identify the 6 people with six points in the plane, $P_1, P_2, P_3, P_4, P_5, P_6$

If two people are friends we will denote this by a line between their respective points and if they are strangers then will be no line connecting their respective points.

Pigeonhole principle implies that, if we consider one of the people, say P_1, then 3 other people will be either friends with P_1 or strangers.

Indeed, we have two holes:

Friend of P_1 Stranger to P_1

and 5 pigeons, the 5 other people

P_2, P_3, P_4, P_5, P_6.

<u>Case 1</u>. P_1 is friend with three other people, P_i, P_j, P_k.

If P_i, P_j, P_k are mutualy strangers our problem is solved, we have 3 mutual strangers.

If at least two of the P_i, P_j, P_k are friends, since each of them are friends with P_1 we found 3 mutual friends.

Case 2. P_1 is stranger to 3 other people, P_i, P_j, P_k.

If P_i, P_j, P_k are mutual friends then we found 3 mutual friends.

If at least 2 of the P_i, P_j, P_k are strangers, since they are strangers to P_1 we have found 3 mutual strangers.

3. In any cocktail party with two or more people, there must be at least two people who have the same number of friends.

Solution Assume there are n people.

Case 1 Assume every one has at least a friend.

Then the holes, are the possible number of friends one might have

| 1 friend | | 2 friends | ------ | n − 1 friends |

We have n pigeons, the n people attending the party.

Then the pigeonhole principle implies that at least two people will be in the same hole, i.e., will have the same number of friends.

Case 2. If there is a person with no friends.

Then either there is another person with no friends which shows two persons with the same number of friends (zero in this case) or all the other people have at least one friend.

But we also know that all the other people cannot be friends with the person with no friends, so they can have at most $n-2$ friends.

Thus in this case the holes are

| 1 friend | | 2 friends | ---- | n-2 friends |

and we have $n-2$ (excluding the one with no friends) pijeons, people

Thus the pijeonhole principle implies that there will be at least two people in the same hole, i.e., with the same number of friends. in this case to.

4. The points of the plane are colored with k colors. Show that there exists at one rectangle with its vertices of the same color.

Solution Assume $k=2$. Then extend the proof to general k. Consider three points P_1, P_2, P_3 a line L parallel to the y axis.
Then the pijeonhole principle implies that two points will have the same color

(2)

In how many ways can we color 3 points with two colors. ??

This is given by the number of functions f, between the set of points and the set of colors.

$$f: \{P_1, P_2, P_3\} \longrightarrow \{\text{Color } 1, \text{ Color } 2\}$$

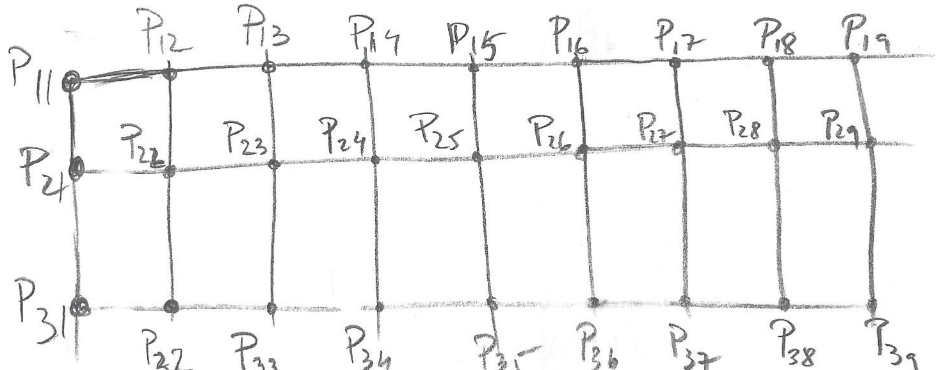
The number of such maps is $2^3 = 8$

Consider 9 triples of 3 points $\{P_{1i}, P_{2i}, P_{3i}\}$

$P_{1i} = P_1 + (i, 0)$, $i = 0, 8$

$P_{2i} = P_2 + (i, 0)$, $i = 0, 8$

$P_{3i} = P_3 + (i, 0)$, $i = 0, 8$



Since we have $2^3 = 8$ colorings of every set of 3 points, and a total of 9 such triplets (P_{1i}, P_{2i}, P_{3i}) then this implies that

two triplets of points will have the same coloring (2)

(2) and (1) imply the statement.

Please extend the proof to $k \geq 3$ colors.

Consider problems from the book

Problem 41 | page 13 (Hint: Try it for $m=2$, $m=3$ and then prove it in general)

Problem 42 | page 13 : Related to problem 3 above

Problem 50 | page 15 : Related to problem 4 above.