# William Lowell Putnam Mathematical Competition
# Problems A1, 1985-2022 (with 2022-2010 first)

A1 ('22) Determine all ordered pairs of real numbers $(a, b)$ such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

A1 ('21) A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

A1 ('20) How many positive integers $N$ satisfy all of the following three conditions?

(i) $N$ is divisible by 2020.

(ii) $N$ has at most 2020 decimal digits.

(iii) The decimal digits of $N$ are a string of consecutive ones followed by a string of consecutive zeros.

A1 ('19) Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where $A$, $B$, and $C$ are nonnegative integers.

A1 ('18) Find all ordered pairs $(a, b)$ of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n + 5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

A1 ('16) Find the smallest positive integer $j$ such that for every polynomial $p(x)$ with integer coefficients and for every integer $k$, the integer

$$p^{(j)}(k) = \left.\frac{d^j}{dx^j} p(x)\right|_{x=k}$$

(the $j$-th derivative of $p(x)$ at $k$) is divisible by 2016.

A1 ('15) Let $A$ and $B$ be points on the same branch of the hyperbola $xy = 1$. Suppose that $P$ is a point lying between $A$ and $B$ on this hyperbola, such that the area of the triangle $APB$ is as large as possible. Show that the region bounded by the hyperbola and the chord $AP$ has the same area as the region bounded by the hyperbola and the chord $PB$.

A1 ('14) Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

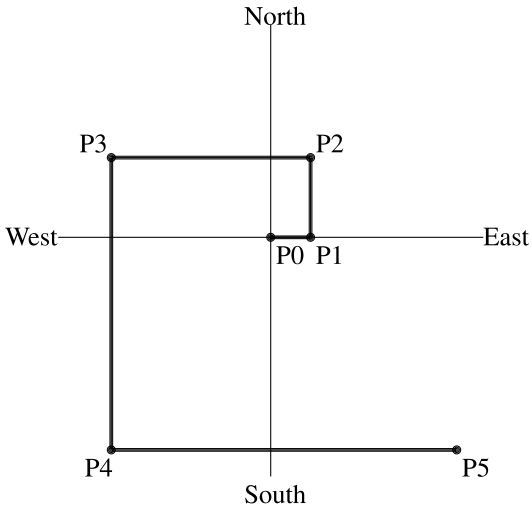
about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

A1 ('13) Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

A1 ('12) Let $d_1, d_2, \dots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices $i, j, k$ such that $d_i, d_j, d_k$ are the side lengths of an acute triangle.

A1 ('11) Define a growing spiral in the plane to be a sequence of points with integer coordinates $P_0 = (0, 0), P_1, \dots, P_n$ such that $n \geq 2$ and:

- The directed line segments $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ are in successive coordinate directions east (for $P_0P_1$), north, west, south, east, etc.
- The lengths of these line segments are positive and strictly increasing.




How many of the points $(x, y)$ with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ cannot be the last point, $P_n$, of any growing spiral?

A1 ('10) Given a positive integer $n$, what is the largest $k$ such that the numbers $1, 2, \dots, n$ can be put into $k$ boxes so that the sum of the numbers in each box is the same?

[When $n = 8$, the example $\{1,2,3,6\},\{4,8\},\{5,7\}$ shows that the largest $k$ is *at least* 3.]

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A1 ('85) Determine, with proof, the number of ordered triples $(A_1, A_2, A_3)$ of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where $a, b, c, d$ are nonnegative integers.

A1 ('86) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers $x$ satisfying $x^4 + 36 \le 13x^2$.

A1 ('87) Curves $A, B, C$ and $D$ are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\}, \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

A1 ('88) Let $R$ be the region consisting of the points $(x, y)$ of the cartesian plane satisfying both $|x|-|y| \le 1$ and $|y| \le 1$. Sketch the region $R$ and find its area.

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A1 ('90) Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \ge 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

A1 ('91) A $2\times 3$ rectangle has vertices as $(0,0)$, $(2,0)$, $(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the $x$-axis is now back on the $x$-axis.) Find the area of the region above the $x$-axis and below the curve traced out by the point whose initial position is (1,1).

A1 ('92) Prove that $f(n) = 1-n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers $n$;

(ii) $f(f(n+2)+2) = n$ for all integers $n$;

(iii) $f(0) = 1$.

A1 ('93) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find $c$ so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the $y$-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A1 ('95) Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any *three* (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication.

A1 ('96) Find the least number $A$ such that for any two squares of combined area 1, a rectangle of area $A$ exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

A1 ('97) A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle $ABC$ has $H$ as the intersection of the altitudes, $O$ the center of the circumscribed circle, $M$ the midpoint of $BC$, and $F$ the foot of the altitude from $A$. What is the length of $BC$?

A1 ('98) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A1 ('99) Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all $x$,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x+2 & \text{if } -1 \le x \le 0 \\ -2x+2 & \text{if } x > 0. \end{cases}$$

A1 ('00) Let $A$ be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

A1 ('01) Consider a set $S$ and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

A1 ('02) Let $k$ be a fixed positive integer. The $n$-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

A1 ('03) Let $n$ be a fixed positive integer. How many ways are there to write $n$ as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with $k$ an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

A1 ('05) Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

A1 ('06) Find the volume of the region of points $(x, y, z)$ such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

A1 ('07) Find all values of $\alpha$ for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

A1 ('08) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers $x$, $y$, and $z$. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers $x$ and $y$.

A1 ('09) Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?