William Lowell Putnam Mathematical Competition
Problems A2, 1985-2022 (with 2022-2010 first)

A2 ('22) Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?

A2 ('21) For every positive real number x, let

$$g(x) = \lim_{r \to 0} ((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}.$$

Find $\lim_{x\to\infty} \frac{g(x)}{x}$.

A2 ('20) Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}.$$

A2 ('19) In the triangle $\triangle ABC$, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

A2 ('18) Let $S_1, S_2, \dots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \dots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

A2 ('17) Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

A2 ('16) Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

A2 ('15) Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

A2 ('14) Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

A2 ('13) Let S be the set of all positive integers that are not perfect squares. For n in S, consider choices of integers $a_1, a_2, \dots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

A2 ('12) Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

A2 ('11) Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S.

A2 ('10) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

A2 ('85) Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

A2 ('86) What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor?$$

A2 ('87) The sequence of digits

$$123456789101112131415161718192021\dots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

A2 ('88) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

A2 ('89) Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive.

A2 ('90) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \dots)$?

A2 ('91) A 2×3 rectangle has vertices as $(0, 0), (2, 0), (0, 3)$, and $(2, 3)$. It rotates $90°$ clockwise about the point $(2, 0)$. It then rotates $90°$ clockwise about the point $(5, 0)$, then $90°$ clockwise about the point $(7, 0)$, and finally, $90°$ clockwise about the point $(10, 0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

A2 ('92) Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1 + x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

A2 ('93) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \dots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

A2 ('94) Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

A2 ('95) For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

A2 ('96) Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

A2 ('97) Players $1, 2, 3, \dots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

A2 ('98) Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

A2 ('99) Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^k (f_j(x))^2.$$

A2 ('00) Prove that there exist infinitely many integers n such that $n, n + 1, n + 2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

A2 ('01) You have coins $C_1, C_2, \dots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k + 1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

A2 ('02) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

A2 ('03) Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be nonnegative real numbers. Show that

$$\begin{aligned}&(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n}\\ &\leq [(a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n)]^{1/n}.\end{aligned}$$

A2 ('04) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

A2 ('05) Let $\mathbf{S} = \{(a, b) | a = 1, 2, \dots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \dots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,

(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?

(An example of such a rook tour for $n = 5$ was depicted in the original.)

A2 ('06) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

A2 ('07) Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)

A2 ('08) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A2 ('09) Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.