Cesàro convergence of spherical averages for measure-preserving actions of Markov semigroups and groups

The talk is based on a joint work with A. Bufetov and M. Khristoforov [1].

Consider a semigroup Γ that is Markov with respect to a set of generators O. Recall the definition of a Markov semigroup. The semigroup Γ has a natural norm: $|g|_O$ is a minimal length of a word representing g; denote $S_O(n) = \{g : |g|_O = n\}$. Consider a directed graph $\mathbf{G}$, its vertex v_0 and a map $\xi \colon \mathcal{E}(\mathbf{G}) \to O$ from the set of its edges to the set O. This map is naturally extended to the map $\bar{\xi}$ of the set $\mathcal{P}(v_0)$ of all finite paths in $\mathbf{G}$ starting at v_0, to the group Γ: $\bar{\xi}(e_1 \dots e_n) = \xi(e_1) \dots \xi(e_n)$. The group Γ is called *Markov* with respect to the set O if the map $\bar{\xi}$ is bijective and it maps any path of length n into $S_O(n)$.

Suppose that the semigroup Γ acts on a probability space (X, ν) by measure-preserving transformations T_g, $g \in \Gamma$. Take any function $\varphi \in L^1(X, \nu)$ and consider the sequence of its *spherical averages*:

$$s_n(\varphi) = \frac{1}{\# S_O(n)} \sum_{g \in S_O(n)} \varphi \circ T_g$$

($\#$ stands for the cardinality of a finite set; if $S_O(n) = \varnothing$, then we set $s_n(\varphi) = 0$). Next, consider the Cesàro averages of the spherical averages:

$$c_N(\varphi) = \frac{1}{N} \sum_{n=0}^{N-1} s_n(\varphi).$$

Theorem 1 *Let Γ be a Markov semigroup with respect to a finite generating set O. Assume that Γ acts by measure-preserving transformations on a probability space (X, ν). Then for any p, $1 \le p < \infty$, and any $\varphi \in L^p(X, \nu)$ the sequence $c_N(\varphi)$ converges in $L^p(X, \nu)$ as $N \to \infty$. If, additionally, $\varphi \in L^\infty(X, \nu)$, then the sequence $c_N(\varphi)$ converges ν-almost everywhere as $N \to \infty$.*

Gromov [3] proves that any hyperbolic group (in the sense of Gromov) is Markov with respect to any symmetric set of generators, thus this theorem can be applied to any hyperbolic group.

In case of irreducible graph $\mathbf{G}$ the theorem is proven by Bufetov [2]. The proof in general case is obtained through a decomposition of the graph $\mathbf{G}$ into smaller blocks (eventually irreducible ones).

References

[1] A. Bufetov, M. Khristoforov, A. Klimenko, Cesàro convergence of spherical averages for measure-preserving actions of Markov semigroups and groups. arXiv:1101.5459v1 [math.DS]

[2] A.I. Bufetov, Markov averaging and ergodic theorems for several operators, in *Topology, Ergodic Theory, and Algebraic Geometry*, *AMS Transl.* **202** (2001), 39–50.

[3] M. Gromov, Hyperbolic groups, in *Essays in Group Theory*, *MSRI Publ.* **8** (1987), 75–263, Springer-Verlag, New York.