April 29, 2025*

The numbers of statements/pages/etc. in the textbook are marked with a †. E.g., Section 1.7† means that 1.7 is the same number as in the textbook, page 70† refers to page 70 of the textbook. Moreover: to make the numbering more useful, theorem 7 in chapter 3 will be labeled 3.7, **not just 7, as in the book.**

Contents

*If you find typos/errors/omissions/etc. please let me know, will correct them for a subsequent edition.

0 Terminology, highlights & connections

0.1 Highlights

Here are some of the main points of this course.

(Zer) Zeros of an analytic function cannot converge to a point where the function is analytic, unless the function is constant zero.

If $f(z)$ is analytic and has a zero of order p at z_0, then $g(z) := f(z)/(z - z_0)^p$ is analytic and nonzero at z_0.

If $f(z)$ has a pole of order p at z_0, then $g(z) := f(z)(z - z_0)^p$ is analytic and nonzero at z_0.

(CR) partial derivatives are continuous around $w \in \mathbb{C}$ and the Cauchy-Riemann equations hold at $w \implies f$ is complex differentiable at $w \implies$ Cauchy-Riemann hold at w

(DefInv) Deformation Invariance Theorem (for contour integrals)

(IntTh) Cauchy's Integral Theorem

(IntFor) Cauchy's Integral Formula

(Harmon) The real (and therefore imaginary) parts of an analytic function is harmonic.

Conversely: a harmonic function in a simply-connected domain (e.g., a disk) is the real part of an analytic function

(ResTh) The residue theorem

(LauSer) An analytic (for now, "complex differentiable") function can be expanded in a Laurent series around a "singularity"; if this point is not a true singularity, then the Laurent series is actually a power series[1].

If the Laurent series has no negative powers ($\iff$ the function is bounded near the singularity), then the singularity is removable.

The singularity is a pole if the Laurent series has only finitely many negative powers ($\iff$ the function converges to ∞ at the singularity).

The singularity is an essential singularity if the Laurent series has infinitely many negative powers ($\iff$ the function has no limit at the singularity).

[1] Which is the true meaning of "analytic"; the "complex differentiable" version is more appropriately called "holomorphic", but they coincide in this case.

(LiouTh) Liouville's Theorem: a bounded entire function is constant.

(MaxMod) The Maximum Modulus Principle: if $|f|$, the absolute value of an analytic function f, has a local maximum then the function is constant. So such a *maximum modulus* can happen only on the "boundary" of the domain.[2]

(ArgPrin) The Argument Principle: counting "zeros − poles" within a simple closed contour

(Index) Computing the winding number of a closed curve about a point

(Rouché) Invariance of the "number of zeros − number of poles" under small perturbations

How these are related:

- DefInv $\iff$ IntTh $\Longrightarrow$ IntFor $\Longrightarrow$ LauSer
- LauSer $\Longrightarrow$ Zer
- CR $\Longrightarrow$ Harmon
- IntFor $\Longrightarrow$ LiouTh
- IntFor $\Longrightarrow$ MaxMod
- IntTh $\Longrightarrow$ ResTh
- ResTh $\Longrightarrow$ ArgPri $\Longrightarrow$ Rouché
- etc.

0.2 Terminology

Here are some of the terms, with a brief description

- in general z stands for a complex number, $z \in \mathbb{C} \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; $z = x + iy$ denotes its real and imaginary parts (so x, y are real numbers).
- for a complex-valued function we often denote $f = u+iv$ for its real and imaginary components
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk in $\mathbb{C}$.
- $\mathrm{Arg}(z) \in (-\pi, \pi]$ denotes the principal value of $\arg(z)$, for $z \neq 0$.
- $f(z)$ is differentiable at z_0 means that f has a (complex) derivative at $z_0 \in \mathbb{C}$
- $f(z)$ is analytic at z_0 it is differentiable on a neighborhood of z_0
- $f : \mathbb{C} \to \mathbb{C}$ is an entire function if it analytic on all of $\mathbb{C}$
- a singular point (a.k.a. singularity) of f is a point where f is not analytic but is the limit of points where f is analytic; e.g. 0 is a singularity of $f(z) = 1/z$; see page 70†
- a singularity is removable if can redefine the function at that singular point to make it analytic; e.g. 1 for $(z^2 - 1)/(z - 1)$, which equals $z + 1$ except at $z = 1$.

[2] Used quotation marks for "boundary" because if the domain is unbounded then there need not be a point where the modulus has an extreme value.

- $\mathrm{Log}(z)$ stands for the principal value of the complex logarithm $\log(z)$ which extends the real logarithm, denoted $\mathrm{Log} = \mathrm{Log}_{\mathbb{R}} : (0, \infty) \to \mathbb{R}$ (see section 3.3†). Then

$$\log(z) := \mathrm{Log}\,|z| + i \arg(z) \qquad \mathrm{Log}(z) := \mathrm{Log}\,|z| + i\,\mathrm{Arg}(z)$$

- D^* in sections 3.3†and 3.5†denotes $\mathbb{C}$ without the real semi-axis $(-\infty, 0]$.
- smooth arc: subset Γ in $\mathbb{C}$ that has an admissible parametrization, that is $z : [a, b] \to \Gamma$ such that
  - z is continuously differentiable
  - its derivative z' is never zero
  - z is one-to-one
- smooth closed curve: same as above but $z(a) = z(b)$ and 1-1 otherwise
- smooth contour: concatenation of smooth curves (need not be closed)
- loop: closed smooth contour
- $\mathrm{Res}(f; a)$, the residue of $f(z)$ at a: the coefficient of $(z - a)^{-1}$ in the Laurent series of f centered at a

1 Chapter 1†: Complex Numbers

1.1 The Algebra of Complex Numbers
1.2 Point Representation of Complex Numbers
1.3 Vectors and Polar Forms
1.4 The Complex Exponential
1.5 Powers and Roots
1.6 Planar Sets
1.7 The Riemann Sphere and Stereographic Projection

1.7 Section 1.7†: The Riemann sphere and stereographic projection

If $z = a + ib \in \mathbb{C}$ corresponds to (x_1, x_2, x_3) on the Riemann (unit) sphere, then

$$x_1 = \frac{2a}{a^2 + b^2 + 1}, x_2 = \frac{2b}{a^2 + b^2 + 1}, x_3 = \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}$$

and, provided (x_1, x_2, x_3) is not the North pole $(0, 0, 1)$,

$$a = \frac{x_1}{1 - x_3}, b = \frac{x_2}{1 - x_3}$$

Without proof: the stereographic projection preserves angles (i.e., is is *conformal*) and takes circles on the Riemann sphere to circles or lines in the complex plane (see Figure 1.23†).

2 Chapter 2†: Analytic Functions

2.1 Functions of a Complex Variable
2.2 Limits and Continuity
2.3 Analyticity
2.4 The Cauchy-Riemann Equations
2.5 Harmonic Functions

2.2 Section 2.2†: Limits and Continuity

Note the definition of convergence to ∞ on page 62†: $z_n \to \infty \iff |z_n| \to \infty$, and similarly $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} |f(z)| = \infty$. See problem 23†: convergence to infinity corresponds via the stereographic projection to convergence to the North pole.

2.3 Section 2.3†: Analyticity

Definition 2.1 (See e.g. top of page 70†)

f is *analytic* at a point z_0 if is differentiable on a neighborhood of z_0.

f is *entire* if it is analytic on the whole complex plane.

A *singular point (a.k.a. singularity)* is a point where f is not analytic but which is the limit of points where f is analytic.

E.g., the complex exponential is an entire function.

2.4 Section 2.4†: The Cauchy-Riemann Equations

"Cauchy-Riemann equations" is usually abbreviated CR. For $f = u + iv$ they are:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Remark 2.1 (How to remember the Cauchy-Riemann equations)

- Require that the derivatives of $u + iv$ in the horizontal and vertical directions (i.e, w.r.t x and iy) be equal (see top of page 74†).
- Another explanation:
  - Multiplication by a complex number $z = a + ib$ on $\mathbb{C}$ is a linear transformation $\xi \in \mathbb{C} \mapsto z\xi \in \mathbb{C}$. Seeing $\mathbb{C} = \mathbb{R}^2$, this linear transformation has a 2×2 matrix, which is (recall the linear algebra course)

    $$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{2.1}$$

    This gives a *representation* of $\mathbb{C}$ as 2×2 matrices with real entries, consistent with both addition and multiplication in $\mathbb{C}$ (that is, $\mathbb{C}$ is isomorphic to this set of 2×2 matrices)
  - CR at z_0 for $f(x + iy) = u + iv \iff$ its *Jacobian matrix* at z_0 (that is, the matrix of partial derivatives),

    $$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

    is of the form (2.1).

Theorem 2.2

- *If $f = u + iv$ is (complex) differentiable at z_0, then CR holds at z_0.*
- *Conversely, if CR holds at z_0 and the partial derivatives are continuous on a neighborhood of z_0, then f is (complex) differentiable at z_0.*

2.5 Section 2.5†: Harmonic functions

Harmonic functions are solution to the Laplace equation; they appear in many physical phenomena.

Theorem 2.3

- *If f is complex-analytic, then its real part (and therefore its imaginary part as well) are harmonic.*
- *Conversely, if u is harmonic on a simply connected domain*[3] *G then there is a function v on G such that $u + iv$ is complex-analytic; v is called the harmonic conjugate of u.*

3 Chapter 3†: Elementary Functions

3.1 Polynomials and Rational Functions
3.2 The Exponential, Trigonometric and Hyperbolic Functions
3.3 The Logarithmic Function
3.4 Washers, Wedges, and Walls
3.5 Complex Powers and Inverse Trigonometric Functions

3.3 Section 3.3†: The Logarithmic Function

Obtain, from solving $w = \log(z) \iff \exp(w) = z$, that

$$\log(z) = \operatorname{Log}|z| + i\arg(z), \qquad \text{for } z \neq 0$$

where the LHS is the (multivalued) complex logarithm and the RHS uses the "real" logarithm (defined only for positive real numbers).

Denote

$$\operatorname{Log}(z) = \operatorname{Log}|z| + i\operatorname{Arg}(z), \qquad \text{for } z \neq 0$$

for the principal value of the logarithm. Recall that $\operatorname{Arg}(z) \in (-\pi, \pi]$ for $z \neq 0$.

Note that $\operatorname{Log}(z)$ extends the real logarithm from $(0, \infty)$ to $\mathbb{C}\backslash\{0\}$ and is analytic on $\mathbb{C}\backslash(-\infty, 0]$.

To compute the derivative of the complex log we used this theorem (see also footnote on page $121^\dagger$):

Theorem 3.1 *Let $f : G \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the open set G, $z_0 \in G$, and denote $w_0 = f(z_0)$.*

If $f'(z_0) \neq 0$ then f is locally invertible at z_0 (that is, there is an open disk B centered at z_0, $z_0 \in B \subset G$, such that $f : B \to f(B) \subset \mathbb{C}$ is invertible), and the inverse is also (complex) analytic.

Therefore, by the chain rule (where, abusing notation, we denote this inverse by f^{-1}),

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

3.4 Section 3.4†: Washers, Wedges, and Walls

Did not discuss this yet, will return to it when needed.

[3] That is, a domain that has "no holes". E.g., $\mathbb{C}$ and $\mathbb{D}$ are simply connected, but $\mathbb{D}$ without the origin or an annulus are not simply connected.

3.5 Section 3.5†: Complex Powers and Inverse Trigonometric Functions

Define

$$z^\alpha := \exp(\alpha \log(z)) = \exp[\alpha(\operatorname{Log}|z| + i\arg(z))] \qquad \text{for } z \neq 0 \text{ and } \alpha \in \mathbb{C}.$$

Note that this is a multivalued function (with finitely many values if α is real and rational – in which case it gives exactly the roots discussed in Section 1.5†).

4 Chapter 4†: Complex Integration

Idea: *Given a (piecewise smooth oriented) contour* $\Gamma \subset \mathbb{C}$ *parameterized by* $\gamma : [a, b] \to \Gamma$, *define* [4] *the* contour integral *on* Γ *of* $f : \Gamma \to \mathbb{C}$ *by* [5]

$$\int_\Gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt \in \mathbb{C}$$

Facts:

- *because* $f(z), \gamma'(t) \in \mathbb{C}$, *can take their product* [6]
- *the integral does not depend on the piecewise smooth parametrization chosen*
- *the length of* Γ *is given by (we assume* $a \leq b$, *to get a non-negative value)*

$$\text{length}(\Gamma) = \ell(\Gamma) = \int_a^b |\gamma'(t)|dt$$

and is also independent of the parametrization.

4.1 4.1†: Contours

Rule: for closed simple curves, orientation is such that **the interior is on the left**.

That there is an "interior" follows from the Jordan Curve Theorem (see Theorem 1† in the book).

4.2 4.2†: Contour Integrals

To simplify our approach, we ***define*** the contour integral according to the formula derived in Theorem 4†. Then Corollary 1† has to be **proven** from our definition (not very difficult).

[4] This is not what the book does: there the integral is defined as a limit of Riemann sums, and then formula (4.1) is a Theorem.

[5] This is similar to a line integral in Calculus III.

[6] For line integrals in $\mathbb{R}^n$ had to use the inner product of two vectors, so the outcome was a real number.

Definition 4.1 Given $\Gamma \subset \mathbb{C}$ a smooth contour and $f : \Gamma \to \mathbb{C}$ a function we ***define***

$$\int_\Gamma f(z)dz := \int_a^b f(\gamma(t))\,\frac{d\gamma}{dt}(t)dt \tag{4.1}$$

for an admissible parametrization of Γ. See Definition $1^\dagger$ of §$4.1^\dagger$ for what *admissible parametrization* means. [Fact: the integral does not depend on the admissible parametrization chosen, as Corollary $1^\dagger$ says.]

We computed in class a few examples:

Example 4.2 For n integer,

$$\int_{|z|=1} z^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$$

by using the parametrization $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Example 4.3

$$I = \int_{|z|=1} \frac{1}{z(2z-1)} dz = 0$$

We computed this by decomposing the rational function in simple fractions,

$$f(z) := \frac{1}{2}\frac{1}{z(z-1/2)} = -\frac{1}{z} + \frac{1}{z-1/2}$$

and then integrating each term separately. The first integral gives $-2\pi i$ (can redo Example 4.2), for the second we get, using the parametrization $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$ for the unit circle:

$$I_2 := \int_{|z|=1} \frac{1}{z-1/2} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}-1/2} dt = \text{"}\log(e^{it}-1/2)|_0^{2\pi}\text{"}$$

and can use the anti-derivative[7] $F(t) = \log(e^{it} - 1/2)$, provided we take a ***continuous branch*** of this multivalued function. Doing this shows that if we start with $F(0) = \log_{\mathbb{R}}(1/2)$ then, ***going around the origin*** along the circle $t \mapsto e^{it} - 1/2$, so counterclockwise, $F(2\pi^-) = \log_{\mathbb{R}}(1/2) + 2\pi i$. Thus $I_2 = 2\pi i$.

We used here $\log_{\mathbb{R}}$ to denote the "real-valued" logarithm defined on $(0, \infty)$ and $F(2\pi^-)$ means the value we get by approaching 2π from the left. Note that $e^{it} - 1/2$ has the same value at $t = 0$ and $t = 2\pi$, namely $1/2$, but $F(0) \neq F(2\pi)$ if we want F to be continuous!

So $I = -2\pi i + 2\pi i = 0$.

NOTE: This can be computed more easily with the Residue Theorem of Chapter $6^\dagger$, according to which for an analytic function having finitely many singularities inside Γ:

$$\int_\Gamma f(z)dz = 2\pi i \sum \{\text{residues of } f \text{ at the singularities inside } \Gamma\}$$

For our f, the singularities inside the unit circle are 0 and $1/2$, and their residues (as seen from the simple fraction decomposition) are -1 and 1.

Example 4.4 Integrate $\overline{z}$ on the contour Γ given by the triangle with vertices 0, 1 and $1 + i$.

For each side: write a parametrization, apply the formula (4.1); then add up the results, and obtain i.

This shows (again) that $f(z) = \overline{z}$ is not an analytic function; if it were analytic, then its integral on a contractible closed contour would be zero, by Cauchy's Integral Theorem, Theorem $4.9^\dagger$.

[7] See e.g. Theorem $6^\dagger$ in §$4.3^\dagger$.

4.3 4.3† Independence of Path

Theorem 4.5 (Theorem 4.7†) *Assume f is continuous on the domain D. TFAE*[8]*:*

(a) f has an antiderivative on D (w.r.t. complex differentiation);

(b) every loop integral of f in D is zero;

(c) contour integrals of f are independent of path (that is, depend only on the initial and final points)

Remark 4.6 In Calculus II the same result was discussed for (real) vector fields: "f has an antiderivative" in (a) is replaced by "the vector field $\mathbf{v}(x, y)$ is a gradient"[9] and the "complex" path-integral is replaced by the "real" path-integral [10].

Note that the "real" version of this result holds in any dimension.

4.4 4.4† Cauchy's Integral Theorem

Theorem 4.7 (Theorem 4.8†: Deformation Invariance Theorem) *Assume the contour Γ_0 can be deformed (continuously) to Γ_1 **inside** the domain D, and f is analytic on D. Then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz$$

Remark 4.8 The ("simple") proof in the textbook relies on a few extra assumptions. For a proof without those, see e.g. [Rud87, Theorems 10.13 and 10.14].

In particular, if D is simply connected then all loop integrals of a function analytic on D are zero (see Theorem 4.9†). Therefore, using Theorem 4.7†:

Theorem 4.9 (Theorem 4.10†) *Assume D is simply connected and f is analytic on D. Then*

f has an antiderivative

which we know (from Theorem 4.7†) is equivalent to

its contour integrals are independent of path

and

loop integrals vanish.

4.5 4.5† Cauchy's Integral Formula and Its Consequences

Theorem 4.10 (Theorem 4.14†: Cauchy's Integral Formula) *Assume f is analytic on the domain D; let $\Gamma \subset D$ be a contour such that its interior is in D (that is, can deform **inside** D the contour Γ to a point). Then*

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz \qquad \text{for each } z_0 \text{ in the interior of } \Gamma$$

[8] "The Following Are Equivalent"

[9] That is, $\mathbf{v} = (\varphi_x, \varphi_y)$ for a real-valued function $\varphi(x, y)$, called potential.

[10] Which uses dot-product instead of the complex multiplication

This is one of the **central** facts about analytic functions, from which much of the rest follows. More generally (collecting a few theorems in the book):

Theorem 4.11 (Theorem 4.15†) *Assume Γ is a simple (so without self-intersections) closed contour in $\mathbb{C}$, and $g : \Gamma \to \mathbb{C}$ is a continuous function. Define*

$$G(z) := \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta \qquad \textit{for } z \textit{ inside } \Gamma.$$

Then G is analytic inside Γ, and has all (complex) derivatives, which are given by

$$\frac{d^k G}{dz^k}(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

That is, "can differentiate under the integral sign".

Remark 4.12 It does not follow that the limit of $G(z)$ as z approaches $z_0 \in \Gamma$ equals $g(z_0)$. For example, take $g(z) = 1/z$ and Γ the unit circle, which gives $G(z) = 0$ inside the unit disk.

Consequences:

Theorem 4.13 (Theorem 4.16†) *An analytic function is infinitely differentiable. That is, all its derivatives are analytic (meaning that they are complex differentiable).*

[Will see later that, even better, an analytic function can be written locally as a power series[11].]

Theorem 4.14 *Let f be a function analytic on and inside the closed simple contour Γ and z inside Γ. Then*

$$\frac{d^k f}{dz^k}(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

Proof: This is a consequence of Theorem 4.15†, and the Cauchy Representation Formula, Theorem 4.14†.

Note that this is a "special case" of the Residue Theorem stated in Chapter 6. ∎

Theorem 4.15 (Theorem 4.18†, Morera) *If all loop integrals of a continuous function inside a domain D are zero, then the function is analytic.*

Proof: We saw that the vanishing of the integrals on all loops is equivalent to having an antiderivative, so it is the derivative of an analytic function. By the above, all derivatives of an analytic function are analytic. ∎

4.6 4.6† Bounds for Analytic Functions

Two main results here: Liouville's theorem and the Maximum Modulus Principle (the latter has versions in other areas of mathematics as well).

Theorem 4.16 (Liouville) *A bounded entire function is constant.*

[11] Actually, this is what "analytic" really means, for both real and complex functions. Complex differentiable on an open set in $\mathbb{C}$ is better called "holomorphic". These two notions are equivalent, that is why the book uses "analytic" only.

An application is the Fundamental Theorem of Algebra: any polynomial of degree $n \geq 1$ with complex coefficients has at least one (and hence n, if we count multiplicities) complex roots. Idea of proof: assume by contradiction that $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with $n \geq 1, a_n \neq 0$, has no roots in $\mathbb{C}$; let $f(z) := 1/P(z)$. Then f is entire (easy) and bounded (not too complicated); by Liouville's theorem, P is constant, contradiction.

Other applications are in the exercises.

The other main result is the Maximum Modulus Principle. There are a few theorems in the book stating various versions, here is one way to think about it:

Theorem 4.17 (Maximum Modulus Principle) *Assume f is analytic on the domain D.*

(a) If $|f|$ has a local maximum inside D, then f is constant on D.

(b) If f extends continuously to $\overline{D}$ [12] *for D a bounded domain and $|f|$ has a maximum on $\overline{D}$ then*

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|$$

where ∂D is the boundary of D. That is, the maximum has to occur on the boundary (which is satisfied for constant functions).

5 Chapter 5†: Series Representations for Analytic Functions

Idea: *Functions that are complex differentiable on an open set are more precisely called* **holomorphic**; *functions that can be represented by a power series (which must be their Taylor series, see more below) are called* **analytic**.

Analytic functions are "as good as it gets", can work with power series as with polynomials.

MAIN POINT: *holomorphic functions (so those that have a complex derivative on an open set) are analytic (can be written as a power series around each point).*

5.1 Review from Calculus II; more details in the book

- $\sum_k |a_k| < \infty \implies \sum_k a_k$ converges (absolute convergence implies convergence; same proof as in the real case)
- the root test, relation to the ratio test

[12] $\overline{D}$ denotes the closure of D

- radius of convergence for a power series:

  Consider the (formal) power series $\sum_{n=0}^{\infty} a_n z^n$. Define its ***radius of convergence*** by

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

  Assume $R > 0$. Then:

  (a) the series converges for $|z| < R$; denote by $f(z)$ its limit
  (b) the series diverges for $|z| > R$
  (c) if $0 < r < R$ then the series converges uniformly on $|z| \le r$
  (d) the function $f(z)$ is infinitely differentiable on $|z| < R$
  (e) the coefficients are given by the derivatives of f:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

- In particular, if a_n are all nonzero for large n and $|a_{n+1}/a_n| \to L$ as $n \to \infty$ then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ (think about the geometric series, $a_n = a^n$), so $R = 1/L$.

- operations with power series: $f+g$, $f \cdot g$, f/g, e^f;

  radius of convergence "as expected";

  examples: $e^z \sin(z)$, $\tan(z) = \sin(z)/\cos(z)$, $e^{\sin z}$, $\log(1+z)$ (convergent for $|z| < 1$ only, b/c $\log(0)$ not defined)

- can differentiate and integrate a power series term-wise, radius of convergence does not change

- in particular, a power series is holomorphic in the interior of its domain of convergence (always an open disk)

Examples 5.1 Power series computations.

(a) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, obtained by computing the derivatives of e^z at $z = 0$ (recall that $0! = 1$).

  Radius of convergence: $a_n = \frac{1}{n!}$, so $\frac{a_{n+1}}{a_n} \to 0$ and therefore $\sqrt[n]{a_n} \to 0$, which gives $R = 1/0 = \infty$. Indeed, this is an entire function.

(b) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$ because $1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1-z}$ (multiply by $(1-z)$ to check). Note that the series does not converge for any $|z| \ge 1$.

(c) $\operatorname{Log}(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$ for $|z| < 1$.

  Radius of convergence: $|a_n| = \frac{1}{n}$ so $|a_{n+1}/a_n| \to 1$, therefore $\sqrt[n]{|a_n|} \to 1$ as well, and then $R = 1$.

  Note that $\operatorname{Log} z$ is not defined for $z = 0$, so the radius of convergence cannot be more than 1.

To compute: either compute the derivatives of $\mathrm{Log}(1+z)$ at $z=0$ or notice that $\mathrm{Log}(1+z)' = \frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n$ for $|z|<1$ from above, and integrate this power series to get

$$C + \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} z^{n+1}$$

To find C, compute $\mathrm{Log}(1+z)$ at $z=0$.

BTW, the above series converges for $z=1$ (being the alternate harmonic series, and indeed $\mathrm{Log}\,2$ exists) but diverges at $z=-1$ (being the harmonic series), consistent with the fact that $\mathrm{Log}(0)$ does not exist.

(d) $\dfrac{e^z}{1-z}$
To obtain the first few terms of the series around $z=0$: instead of computing the derivatives at zero, can multiply (as for polynomials) the series of e^z and the series of $\frac{1}{1-z}$

(e) the series around zero of $\dfrac{z^2+z+1}{1-z}$ can be obtained by expanding $(z^2+z+1)(\sum_{n=0}^{\infty} z^n)$, instead of computing derivatives. The radius of convergence cannot be more than 1 (because there is a singularity at $z=1$), and for $|z|<1$ we use the correct expansion of $1/(1-z)$. The result will also have radius of convergence equal to 1.

(f) and so on

5.2 Main Results from 5.2†-5.5†

Theorem 5.2 (Theorem 5.3†, in §5.2†) *If f is analytic (meaning "holomorphic") on the open disk $D_R := \{|z-z_0|<R\}$ then on D_R the function f is equal to its Taylor series centered at z_0 (which means that f is "analytic" on D_R, see above discussion):*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \qquad \text{for } |z-z_0|<R$$

In particular, the radius of convergence of this series is at least R.

Recall that, as a consequence of the Cauchy Integral Formula,

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\, d\zeta \qquad k \geq 0$$

where C is any simple positively oriented curve encircling z_0 inside D_R.

Theorem 5.3 (Theorem 5.14†, in §5.5†) *If f is analytic (meaning "holomorphic") on the open annulus $A = \{r<|z-z_0|<R\}$ then on A the function f can be written as a* **Laurent series** *centered at z_0:*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k \qquad \text{for } r<|z-z_0|<R$$

In particular, the "power series" converges for $|z-z_0|<R$, the "inverse-power series" converges for $|z-z_0|>r$.

The coefficients are given by

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta \qquad k \in \mathbb{Z}$$

where C is any simple positively oriented curve inside the annulus encircling z_0.

Examples

- the power series of
  - e^z, defined as $e^{x+iy} = e^x(\cos y + i\sin y)$
  - $\cos z$, defined as $(e^{iz} + e^{-iz})/2$
  - $\sin z$, defined as $(e^{iz} - e^{-iz})/(2i)$
  - $\log(1+z)$ for $|z| < 1$ (recall that different branches of log differ by a multiple of $2\pi i$)

  are as for their real versions.

- the function $f(z) := \dfrac{z^2}{(z-1)^3(z-2)^2}$ or $g(z) := \dfrac{e^{\sin z}}{(z-1)^3(z-2)^2}$
  - can be written as a ***power series*** on disks that do not contain the points 1 and 2 (and the radius of converge is limited to not including these singularities)
  - can be expanded ***only in a Laurent series*** on annuli that surround these singularities (need not be centered at a singularity), for example $\{0 < |z-2| < 1\}$ or $\{1 < |z-3| < 2\}$ (with the inner and outer radii also restricted by not including the singularities).

  Actually, the Laurent series for f and g have only finitely many negative powers because both have *poles of finite order* at their singularities (more about this later).

The above theorems follow from the Cauchy Integral Formula, using the geometric series:

Theorem 5.4 (The geometric series) *The function $f(z) = 1/(1-z)$ has the power series expansion*

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \text{for } |z| < 1$$

and the convergence is uniform on each disk $|z| < r < 1$, meaning that for each $\varepsilon > 0$ there is an $N = N_{\varepsilon,r}$ such that

$$\left|\frac{1}{1-z} - \left(\sum_{k=0}^{n} z^k\right)\right| \le \varepsilon \quad \text{for } n \ge N,\ |z| < r$$

Proof: Use the identity $1 + z + z^2 + \cdots + z^n = \dfrac{1-z^{n+1}}{1-z}$ for $z \neq 1$. ∎

Note also that uniform limit of analytic functions is also analytic:

Theorem 5.5 (Theorem 5.9†, §5.3†) *Let $D \subset \mathbb{C}$ be an open set and $f_n : D \to \mathbb{C}$ analytic functions.*

If f_n converges uniformly[13] on D to f, then f is also analytic.

[13] Can relax uniform convergence on the whole D to uniform convergence on each closed bounded subset in D, called ***uniform convergence on compacts*** and more suitable for complex analysis.

Proof: Using Morera's Theorem 4.18†: it suffices to prove that the integral of f on any closed contour $\Gamma \subset D$ is zero; but the integral of f_n on Γ vanishes because f_n is analytic, and – due to the uniform convergence of f_n to f (see Theorem 5.8†) – $\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\,dz$. ∎

Remark 5.6 The above theorem is ***VERY FALSE*** for real-valued analytic functions: any continuous function on $[a, b] \subset \mathbb{R}$ is the uniform limit on $[a, b]$ of polynomials, so of real-analytic functions (this is the Weierstrass approximation theorem).

Examples

If possible, use the expansion of $1/(1-u) = \sum_{k=0}^{\infty} u^k$ for $|u| < 1$. For a rational function, decompose in partial fractions.

Example 5.7 Compute the power/Laurent series centered at zero for $f(z) = \frac{1}{z-1}$. Do it for the disk $\{|z| < 1\}$ and for the annulus $\{|z| > 1\}$.

The computations can be done using the geometric series, no need to compute integrals/derivatives.

Note that the Laurent series centered at $a \in \mathbb{C}$ in a domain that contains ∞ should contain only negative powers, starting with a term of order $1/(z-a)$. This is because as $z \to \infty$, $f(z) \approx 1/z$.

Example 5.8 Compute the power/Laurent series for $f(z) = \frac{1}{(z-1)(z-3)}$. The computations can be done using the geometric series. Do it for disks/annuli centered at 0, with radii 1, and 3; there are 3 regions to consider.

Note that the Laurent series in a domain that contains ∞ should contain only negative powers, starting with terms of order $1/(z-a)^2$. This is because as $z \to \infty$, $f(z) \approx 1/z^2$, so the "leading" term should be $1/z^2$ if the series is centered at zero.

Example 5.9 Compute the power/Laurent series for $f(z) = \frac{1}{(z-1)(z-2)}$ (continuing example 2 in the book). Do it for disks/annuli centered at 1, 3/2, 3.

Note that the Laurent series in a domain that contains ∞ should contain only negative powers, starting with a term of order $1/(z-a)^2$. This is because as $z \to \infty$, $f(z) \approx 1/z^2$.

More about this later: the points 1 and 2 are *poles* for f, both of order one; ∞ is a zero of second order.

Example 5.10 Compute the Laurent series at zero for $f(z) = \exp(1/z)$ (example 4 in the book). Can use the power series of the exponential.

More about this later: the origin is an *essential singularity* for $f(z)$.

Fact: as $z \to 0$, $\exp(1/z)$ takes on any value in the complex plane except 0 (because the exponential is never zero). This is the behavior near and essential singularity, see ***Picard's Theorem*** (Theorem 17† in §5.6†).

Example 5.11 Compute the Laurent series for $f(z) = \frac{1}{(z-1)^2(z-2)}$ on the annulus $\{|z-3/2| > 1/2\}$ centered at 3/2.

The result (after a lengthier computation) is

$$f(z) = \sum_{n\geq 2} \left[1 + (-n+2)(-1)^{n-2}\right] \frac{1}{2^{n-2}} \cdot \frac{1}{(z-3/2)^n} = \frac{1}{(z-3/2)^2} + \dots$$

which starts the right way because $f(z) \approx \frac{1}{z^2}$ near ∞.

Method: first decompose

$$f(z) = \frac{-2}{z-1} + \frac{-1}{(z-1)^2} + \frac{2}{z-2}$$

and then expand each simple fraction.

Note that, despite the partial fraction decomposition, the leading term is not $-1/(z-3/2)^2$ because the other two terms together have a contribution of the same order.

5.6 5.6† Zeros and Singularities

Recall that a singular point (a.k.a. singularity) of f is a point where f is not analytic but which is the limit of points where f is analytic; e.g. 0 is a singularity of $f(z) = 1/z$; see page 70†.

Definition 5.12 (Zeros) Assume f is analytic at z_0.

Then $f(z)$ has a ***zero of order* m** at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \text{ and } f^{(m)}(z_0) \neq 0$$

[See later: then $f(z) = (z-z_0)^m g(z)$, g analytic at z_0, with $g(z_0) \neq 0$.]

Definition 5.13 (Poles) Assume f is analytic and has an ***isolated singularity***[14] at z_0. Consider the Laurent expansion of $f(z)$ centered at z_0,

$$f(z) = \sum_{-\infty}^{\infty} a_k (z-z_0)^k$$

(a) If $a_k = 0$ for all $k < 0$ then the singularity is ***removable***; can define $f(z_0) = a_0$ to get an analytic function at z_0.

(b) If $a_k = 0$ for all $k < -m$, and $a_{-m} \neq 0$ for some $0 < m \neq \infty$, then the singularity is ***a pole of order* m**. [See later: then $f(z) = g(z)/(z-z_0)^m$, g analytic at z_0, $g(z_0) \neq 0$.]

(c) Otherwise, z_0 ***is an essential singularity***. [See Picard's theorem.]

Some results from this section:

Theorem 5.14 *Assume f is analytic in the domain D. If f has a zero of order p at $a \in D$ then $f(z) = (z-a)^p g(z)$ with g analytic on D and $g(a) \neq 0$.*

Proof: Let $g(z) := f(z)/(z-a)^p$ on $D \setminus \{a\}$, which is analytic on $D \setminus \{a\}$. Use the Taylor series of f at a to conclude that g is analytic at a as well. ∎

Theorem 5.15 *Assume f has an isolated singularity at z_0. TFAE:*

(a) z_0 is a removable singularity.

(b) $f(z)$ is bounded around z_0.

(c) $f(z)$ has a finite limit as $z \to z_0$.

[14] That is, the function is analytic on a punctured disk around the singularity.

Theorem 5.16 *Assume f is analytic in the domain $D \setminus \{a\}$, where $a \in D$. If f is has a pole of order p at a then $f(z) = \frac{g(z)}{(z-a)^p}$ with g analytic on D and $g(a) \neq 0$. [Use the Laurent series of f at a.]*

Theorem 5.17 (Consequence of Theorem 5.14) *If f is not constant zero on a domain D, then each zero of f in D is isolated.*

Proof: Let $a \in D$ be a zero of f. Look at the derivatives $f^{(k)}(a)$, $k \geq 1$.

- If all derivatives are zero, then the power series of f is identically zero, so f is constant zero on the disk in D where this series converges.

  This implies that f is constant zero on D. [15]
- otherwise, there is a $k \geq 1$ such that $f^{(k)}(a) \neq 0$, so a is a zero of finite order. Write f as in Theorem 5.14; since $g(a) \neq 0$, there is no other zero close to a .

∎

Corollary 5.18 *If $f, g : D \to \mathbb{C}$ are analytic and coincide on a set that has a limit point in the domain D, then $f = g$ on D. [Apply the previous theorem to $f - g$.]*

Special case: if $f : D \to \mathbb{C}$ is analytic on the domain D and its zeros have an accumulation point in D, then f is identically zero.

Theorem 5.19 (Picard, Theorem 5.17†) *If w is an essential singularity of f then on any neighborhood of w the values of f cover the whole complex plane, except maybe one value.*

Proof: The proof is above the pay grade for this course.

As an example, look at $f(z) = \exp(1/z)$ and its behavior near $z = 0$. ∎

Corollary 5.20 *Assume a is a singularity of f. TFAE:*

(a) $\lim_{z \to a} f(z) = \infty$;

(b) *a is a pole of f.*

Theorem 5.21 (Casorati-Weierstrass) *Near an essential singularity, the range of the function is dense in $\mathbb{C}$.*

Proof: See exercise 5.6.14†. Assume w is not a limit point of the images through f of a punctured neighborhood of an essential singularity z. Then $g(z) := 1/(f(z) - w)$ is bounded near z, so the singularity is removable. But then $f(z) = w + 1/g(z)$ does not have an essential singularity. ∎

Actually:

Theorem 5.22 (Little Picard Theorem) *If a function $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.*

That is, if f misses two values then it is constant.

[15] This statement requires more details; can use an argument similar to the proof that if the absolute value of an analytic function h has a maximum inside a domain, then h is constant — prove a local result (Corollary 3† to Theorem 16† in §5.6†) and then use path connectedness.

Theorem 5.23 (Great Picard Theorem) *If an analytic function f has an essential singularity at a point w, then on any punctured neighborhood of w , $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often. [This follows from Theorem 5.19, Theorem 5.17†.]*

6 Chapter 6†: Residue Theory

6.1 The Residue Theorem
6.2 Trigonometric Integrals over $[0, 2\pi]$: $\int_0^{2\pi} U(\sin, \cos)$ etc.
6.3 Improper Integrals of Certain Functions over $\mathbb{R}$: $\int_{-\infty}^{\infty} P(x)/Q(x)dx$
6.4 Improper Integrals Involving Trigonometric Functions: $\int_{-\infty}^{\infty} \sin(kx)\frac{P(x)}{Q(x)}dx$
6.5 Indented Contours, e.g. $p.v. \int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx$
6.6 Integrals Involving Multiple-Value Functions, e.g. $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+4)}$
6.7 The Argument Principle and Rouché's Theorem

6.1 6.1† Cauchy's Residue Theorem

Theorem 6.1 (Theorem 6.2†, Cauchy's Residue Theorem) *Assume f is analytic on the simple closed contour Γ, and inside Γ the function f is analytic except for finitely many singular points, $z_1, z_2, \ldots, z_n$. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f; z_k)$$

Theorem 6.2 (Theorem 6.1†, computing residues) *If f has a pole of order m at a, then*

$$\mathrm{Res}(f; a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$$

Remark 6.3 (l'Hôpital to the rescue.) If f has a pole of order 1 then

$$\mathrm{Res}(f; a) = \lim_{z \to a}(z-a)f(z) = \lim_{z \to a} \frac{(z-a)}{1/f(z)}$$

It might be easier to compute the limit (or some re-organized version of it) which is $0/0$, by l'Hôpital.

6.2 6.2† Trigonometric Integrals over $[0, 2\pi]$: $\int_0^{2\pi} U(\sin, \cos)$ etc.

Idea:

- *If integral is over $[0, 2\pi]$: write* sin *and* cos *using exponentials and convert into an integral around the unit circle. Then apply the Residue Theorem.*
- *In some other cases, could relate the desired integral to one over $[0, 2\pi]$.*

$$\begin{aligned}\int_0^{2\pi} U(\cos\theta, \sin\theta)d\theta &= \int_0^{2\pi} U\left(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}\right) d\theta \\ &= \int_{\{|z|=1\}} U\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right) \frac{1}{iz}dz\end{aligned}$$

6.3 6.3† Improper Integrals of Certain Functions over $\mathbb{R}$: $\int_{-\infty}^{\infty} P(x)/Q(x)dx$

Remark 6.4 (WARNING) We are computing $\lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(x)dx$, which is the ***principal value, abbreviated "p.v."*** of the integral. The above limit can exists even if $\displaystyle\int_{-\infty}^{0} f(x)dx$ and $\displaystyle\int_{0}^{\infty} f(x)dx$ do not exist; consider $f(x) = x$.

If the integral converges, then the "p.v." is the same as the "regular" integral.

> **Idea:** *Complete the segment $[-\rho, \rho] \subset \mathbb{R}$ on the real axis to a closed contour, so that the contribution of the added curve goes to zero or can be understood; then apply the Residue Theorem.*

Lemma 6.5 (Lemma 6.1†) *Let $f(x) = P(x)/Q(x)$ be a rational function with* degree $Q \geq$ degree $P + 2$, *and C_ρ^+ is the semicircle of radius ρ in the upper half-plane with center the origin, then*

$$\lim_{\rho\to\infty} \int_{C_\rho^+} f(z)dz = 0.$$

Actually, only need that $|P(z)/Q(z)| \leq C/|z|^2$ for large $|z|$.

Remark 6.6 If $|f(x)| \leq C/|x|^2$ for large $|z|$ (and f is continuous on $\mathbb{R}$) then $\int_{-\infty}^{\infty} f(x)dx$ converges, so it is equal to the principal value.

Example 6.7 $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \frac{\pi}{2}\sqrt{2}$

COMPUTATION:

The rational function $f(z) = 1/(z^4+1)$ satisfies the assumptions of Lemma 6.1†, so can make a closed contour by adding to $[-\rho, \rho] \subset \mathbb{R}$ the semicircle C_ρ^+. Call this closed contour Γ_ρ. So, by the Residue Theorem,

$$\int_{\Gamma_\rho} f(z)dz = 2\pi i \sum_{a \text{ pole inside } \Gamma_\rho} \mathrm{Res}(f; a).$$

As $\rho \to \infty$, we will have to include exactly the poles in the upper half-plane.

The poles are given by $z^4 + 1 = 0$, so roots of order 4 of -1, which are $z_k = \exp(i(\pi + 2\pi k)/4)$, $k = 0, 1, 2, 3$. Only z_0 and z_1 are in the upper half-plane.

Compute the residues: since the roots are simple, all the poles are of order one; $\mathrm{Res}(f, z_k) = \lim_{z\to z_k} f(z)(z - z_k)$, which one can compute with l'Hôpital because it is $0/0$. This gives

$$\mathrm{Res}(f, z_k) = \frac{1}{4z_k^3} = \frac{1}{4}\exp(-3i(\pi + 2\pi k)/4) = \frac{1}{4}\left(\cos(-3(\pi + 2\pi k)/4) + i\sin(-3(\pi + 2\pi k)/4)\right)$$

Therefore, invoking Lemma 6.1† (and a bit of trigonometry):

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\rho\to\infty} \int_{\Gamma_\rho} f(z)dz = 2\pi i\left(\mathrm{Res}(f; z_0) + \mathrm{Res}(f; z_1)\right) = 2\pi i \frac{1}{4} 2i\sin(-3\pi/4) = \frac{\pi}{2}\sqrt{2}$$

CHECKING (as much as possible): First of all, the result is a positive real number, which is good.

Second, $f(x)$ is less than $1/x^4$; can bound f on [0,1] by 1 and on $[1, \infty)$ by $1/x^4$. Therefore, using that f is even:

$$\int_{-\infty}^{\infty} f(x)dx < 2\left(1 + \int_1^{\infty} \frac{1}{x^4}dx\right) = 8/3 \approx 2.66$$

The value we got for the integral is $\frac{\pi}{2}\sqrt{2} \approx \frac{3}{2} \cdot 1.4 = 2.1$.

6.4 6.4† Improper Integrals Involving Trigonometric Functions: $\int_{-\infty}^{\infty} \sin(kx)\frac{P(x)}{Q(x)}dx$

Idea: *Write* sin *and* cos *using exponentials; make a closed contour by using:*

- *a semicircle in the upper half-plane for e^{imz} if $m > 0$ and*
- *a semicircle in the lower half-plane for e^{imz} if $m < 0$.*

Lemma 6.8 (Lemma 6.3†, by Jordan) *Let $f(z) = e^{imz}P(z)/Q(z)$ with P, Q polynomials having* degree $Q \geq$ degree $P + 1$. *Denote C_ρ^+ the semicircle of radius ρ centered at the origin in the upper half-plane and C_ρ^- the semicircle of radius ρ centered at the origin in the lower half-plane. Then:*

$$m > 0 \implies \lim_{\rho\to\infty} \int_{C_\rho^+} f(z)dz = 0.$$

$$m < 0 \implies \lim_{\rho\to\infty} \int_{C_\rho^-} f(z)dz = 0.$$

Actually, only need that $|P(z)/Q(z)| \leq C/|z|$ for large $|z|$.

Remark 6.9 If $f(z) = e^{imz}P(z)/Q(z)$ as above then the integral $\int_{-\infty}^{\infty} f(z)dz$ actually converges (by the Alternating Series Test).

Example 6.10 (Example 6.4.2†) $\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+1}dx = \frac{\pi}{e}$
COMPUTATION:

NOTE: this integral is not absolutely convergent, but it is convergent (using the Alternating Series Test). So no need to restrict to the principal value.

Write $\sin x = (\exp(ix) - \exp(-ix))/(2i)$, which gives two integrals involving $\exp(\pm iz)$ and $f(z) = z/(z^2 + 1)$.

- For the integral of $\exp(iz)f(z)$ complete $[-\rho, \rho]$ to a closed contour by adding C_ρ^+, so will have to take into account the poles in the upper half-plane. The closed contour is oriented correctly (its interior is on the left).

  So need the pole of $\exp(iz)z/(z^2 + 1)$ at i (there are a few constants floating around, should include those too).

  After taking $\rho \to \infty$, using Lemma 6.3† (Lemma 6.8) for the $m > 0$ case and the Residue Theorem, the first integral is $\frac{\pi}{2e}$.

- For the integral of $f(z)\exp(-iz)$ complete $[-\rho, \rho]$ to a closed contour by adding C_ρ^-, so will have to take into account the poles in the lower half-plane. Note that the closed contour has the <u>opposite orientation</u> (its interior is on the right). So have to <u>change the sign</u> of the sum of residues.

  Need the pole of $\exp(-iz)z/(z^2 + 1)$ at $-i$ (should include those pesky constants too).

  After taking $\rho \to \infty$, using Lemma 6.3† (Lemma 6.8) for the $m < 0$ case and the Residue Theorem (recall that have to change the sign), the second integral is $\frac{\pi}{2e}$ as well.

Therefore, the answer is $2\frac{\pi}{2e}$.

<u>CHECKING:</u> we get a real value, which is good. Do not see an easy way to decide whether the value is reasonable, but recall that the Alternating Series Test has an error estimate if truncating the series.

Shortcut: it happens that $x\sin(x)/(x^2+1) = \operatorname{Im}[x\exp(ix)/(x^2+1)]$, so could compute only $\int_{-\infty}^{\infty}$ of this function (for that only need to consider the pole at i), and then take the imaginary part.

<u>**NO SUCH SHORTCUT**</u> if had to compute $\int_{-\infty}^{\infty} x\sin x/(x^2+i)dx$.

6.5 6.5† Indented Contours, e.g. $p.v.\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx$

> **Idea:** *If the singularity is on the contour (e.g., in $\mathbb{R}$) then we indeed might have to compute a principal value (that is, the integral might not converge). For that, will go around the singularity on a small semicircle.*

Lemma 6.11 (Lemma 6.4†) *Assume $f(z)$ has a <u>simple pole</u> at $c \in \mathbb{C}$.*

Denote by T_r the arc of circle of radius r centered at c between the angles $\theta_1 < \theta_2$, so parametrized (counter-clockwise) by $t \in [\theta_1, \theta_2] \mapsto c + r\exp(it)$. Then

$$\lim_{r\to 0^+}\int_{T_r} f(z)dz = i(\theta_2-\theta_1)\operatorname{Res}(f;c)$$

In particular, if $c \in \mathbb{R}$ and S_r is the semicircle in the upper half-plane <u>covered from left-to-right</u> (so clockwise), then

$$\lim_{r\to 0^+}\int_{S_r} f(z)dz = -i\pi\operatorname{Res}(f;a)$$

Example 6.12 (Example 6.5.1†)

$$p.v.\int_{-\infty}^{\infty}\frac{\exp(ix)}{x} = i\pi$$

COMPUTATION:

Note that the integral converges at $\pm\infty$, but NOT at 0; so indeed should compute the principal value.

By Lemma 6.3† (Lemma 6.8) can add the integral on C_ρ^+, whose limit is zero. In order to avoid the singularity of zero, will go around it on a small semicircle S_r of radius r in the upper half-plane centered at the singularity.

We consider the closed contour given by the segments $[-\rho,-r]\cup[r,\rho]\subset\mathbb{R}$, the semicircle S_r from the above Lemma and the C_ρ^+ from before; $f(z)$ has no poles inside this contour, so the integral around the contour is zero. Using Lemma 6.3† (Lemma 6.8) and Lemma 6.4†, we obtain that the p.v. is $\pi i\operatorname{Res}(f;0)$.

6.6 6.6† Integrals Involving Multiple-Value Functions, e.g. $\int_0^{\infty}\frac{dx}{\sqrt{x}(x+4)}$

We discussed in class a few examples from the book. This section is not on the exams.

6.7 6.7† The Argument Principle and Rouché's Theorem

> **Idea:** *One can associate a "winding number" (can call it "index" too) of a closed curve around a point. This can be computed with a contour integral. Since this quantity takes integer values, "small" changes will not affect it.*

Such remarkable properties (connecting seemingly unrelated quantities, maybe deriving certain rigid behaviour) appear in many places. One of the more famous ones is the Atiyah-Singer Index Theorem, see e.g. Wikipedia [16]

Definition 6.13 A function is ***meromorphic*** on a domain if at each point the function is either analytic or has a pole (so no essential singularities).

Remark 6.14 As a consequence of the Weierstrass Product Theorem [Wei], any entire meromorphic function is a quotient of two ***analytic*** entire functions. [Something similar should hold for meromorphic functions on any domain.]

Very similar to rational functions, which are quotients of polynomials.

Theorem 6.15 (Theorem 6.3†) *Given a simple closed curve C and a function f that has no zeros on C, is analytic on C, and meromorphic inside C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f) \tag{6.1}$$

where

- *$N_0(f)$ is the number of zeros of f in D, counting multiplicities;*
- *$N_p(f)$ is the number of poles of f in D, counting multiplicities.*

Actually:

Definition 6.16 (Winding number, a.k.a. index) Given a close contour $\gamma : [a, b] \to \mathbb{C}$ and a point $z_0 \in \mathbb{C}$ that is not on the contour, the ***winding number, a.k.a. index*** of γ about z_0 is given by

$$\operatorname{Ind}_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}dt \tag{6.2}$$

Note that (6.2) is a special case of (6.1).

Why:

$$\frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}dt = \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\text{“}\log_{\mathbb{C}}(\gamma(t)-z_0)\text{”}dt = \frac{1}{2\pi i}\log_{\mathbb{R}}(|\gamma(t)-z_0|)+i\text{“}\arg(\gamma(t)-z_0)\text{”}\big|_{t=a}^b$$

$$= \frac{1}{2\pi i}\left[i\text{“}\arg(\gamma(t) - z_0)\text{”}\right]\big|_{t=a}^b$$

and the latter counts how many turns $\gamma(t) - z_0$ did around the origin, which (after a translation by z_0) is the same as the number of turns that $\gamma(t)$ did around z_0.

Remark 6.17 For a contour $C \subset \mathbb{C}$ parametrized by $\gamma : [a, b] \to \mathbb{C}$,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z) - z_0}dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t)) - z_0}\gamma'(t)dt = \operatorname{Ind}_{z_0}(f \circ \gamma)$$

that is, the winding number of $f(C)$ around z_0.

[16] From Wikipedia: In differential geometry, the Atiyah–Singer index theorem, proved by Michael Atiyah and Isadore Singer (1963), states that for an elliptic differential operator on a compact manifold, the analytical index (related to the dimension of the space of solutions) is equal to the topological index (defined in terms of some topological data). It includes many other theorems, such as the Chern–Gauss–Bonnet theorem and Riemann–Roch theorem, as special cases, and has applications to theoretical physics.

Consequences:

Theorem 6.18 (Corrolary 6.1†) *If f analytic on and inside the simple closed contour C, and $z_0 \notin C$, then*

$$\mathrm{Ind}_{z_0}(f(C)) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-z_0}dz = N_{z_0}(f),$$

the number of solutions inside C to $f(z) = z_0$ (counting multiplicities).

Take $z_0 = 0$ to count the zeros.

Theorem 6.19 (Rouché's Theorem 6.4†) *Assume f and h are analytic inside and on the closed simple contour C. If*

$$|h(z)| < |f(z)| \quad \textit{for all } z \in C$$

then f and $f + h$ have the same number of zeros inside C.

Remark 6.20 Note that the condition is imposed ***only along*** C, but it implies what happens ***inside*** C.

Rouché's Theorem can be used to prove:

Theorem 6.21 (Open Mapping Theorem 6.5†) *If f is analytic and not constant, then it is an open mapping: the image of an open set is open.*

Remark 6.22 One can give another proof of the Fundamental Theorem of Algebra using Rouché: let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients; then $|P(z) - a_n z^n| < |a_n z^n|$ for $|z|$ large enough, so Rouché implies that $P(z)$ and $a_n z^n$ have the same number of zeros in $\mathbb{C}$.

References

[Rud87] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[Wei] Weierstrass Product Theorem. `https://encyclopediaofmath.org/index.php?title=Weierstrass_theorem#Infinite_product_theorem`. Accessed: 2025-04-22.