Since D is a diagonal matrix, the system $y' = Dy$ is easy to solve. Setting

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix},$$

we can rewrite $y' = Dy$ as

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} 2y_1(t) \\ 2y_2(t) \\ 4y_3(t) \end{pmatrix}.$$

The three equations

$$\begin{aligned} y_1' &= 2y_1 \\ y_2' &= 2y_2 \\ y_3' &= 4y_3 \end{aligned}$$

are independent of each other, and thus can be solved individually. It is easily seen (as in Example 3 of Section 5.1) that the general solution to these equations is $y_1(t) = c_1e^{2t}$, $y_2(t) = c_2e^{2t}$, and $y_3(t) = c_3e^{4t}$, where c_1, c_2, and c_3 are arbitrary constants. Finally,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = x(t) = Qy(t) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -c_1e^{2t} & & - \ c_3e^{4t} \\ & -c_2e^{2t} & - \ 2c_3e^{4t} \\ c_1e^{2t} & + \ c_2e^{2t} & + \ c_3e^{4t} \end{pmatrix}$$

yields the general solution of the original system. Note that this solution can be written as

$$x(t) = e^{2t}\left[c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right] + e^{4t}\left[c_3\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right].$$

The expressions in brackets are arbitrary vectors in E_{λ_1} and E_{λ_2}, respectively, where $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus the general solution of the original system is $x(t) = e^{2t}z_1 + e^{4t}z_2$, where $z_1 \in \mathsf{E}_{\lambda_1}$ and $z_2 \in \mathsf{E}_{\lambda_2}$. This result is generalized in Exercise 15.

Direct Sums*

Let T be a linear operator on a finite-dimensional vector space V. There is a way of decomposing V into simpler subspaces that offers insight into the

behavior of T. This approach is especially useful in Chapter 7, where we study nondiagonalizable linear operators. In the case of diagonalizable operators, the simpler subspaces are the eigenspaces of the operator.

Definition. *Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a vector space* V. *We define the* **sum** *of these subspaces to be the set*

$$\{v_1 + v_2 + \cdots + v_k \colon v_i \in \mathsf{W}_i \text{ for } 1 \le i \le k\},$$

which we denote by $\mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_k$ or $\sum_{i=1}^{k} \mathsf{W}_i$.

It is a simple exercise to show that the sum of subspaces of a vector space is also a subspace.

Example 8

Let $\mathsf{V} = \mathsf{R}^3$, let W_1 denote the xy-plane, and let W_2 denote the yz-plane. Then $\mathsf{R}^3 = \mathsf{W}_1 + \mathsf{W}_2$ because, for any vector $(a, b, c) \in \mathsf{R}^3$, we have

$$(a, b, c) = (a, 0, 0) + (0, b, c),$$

where $(a, 0, 0) \in \mathsf{W}_1$ and $(0, b, c) \in \mathsf{W}_2$. ◆

Notice that in Example 8 the representation of (a, b, c) as a sum of vectors in W_1 and W_2 is not unique. For example, $(a, b, c) = (a, b, 0) + (0, 0, c)$ is another representation. Because we are often interested in sums for which representations are unique, we introduce a condition that assures this outcome. The definition of *direct sum* that follows is a generalization of the definition given in the exercises of Section 1.3.

Definition. *Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a vector space* V. *We call* V *the* **direct sum** *of the subspaces $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ and write $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$, if*

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$$

and

$$\mathsf{W}_j \cap \sum_{i \ne j} \mathsf{W}_i = \{0\} \quad \text{for each } j \ (1 \le j \le k).$$

Example 9

Let $\mathsf{V} = \mathsf{R}^4$, $\mathsf{W}_1 = \{(a, b, 0, 0) \colon a, b, \in R\}$, $\mathsf{W}_2 = \{(0, 0, c, 0) \colon c \in R\}$, and $\mathsf{W}_3 = \{(0, 0, 0, d) \colon d \in R\}$. For any $(a, b, c, d) \in \mathsf{V}$,

$$(a, b, c, d) = (a, b, 0, 0) + (0, 0, c, 0) + (0, 0, 0, d) \in \mathsf{W}_1 + \mathsf{W}_2 + \mathsf{W}_3.$$

Thus

$$\mathsf{V} = \sum_{i=1}^{3} \mathsf{W}_i.$$

To show that V is the direct sum of W_1, W_2, and W_3, we must prove that $\mathsf{W}_1 \cap (\mathsf{W}_2 + \mathsf{W}_3) = \mathsf{W}_2 \cap (\mathsf{W}_1 + \mathsf{W}_3) = \mathsf{W}_3 \cap (\mathsf{W}_1 + \mathsf{W}_2) = \{0\}$. But these equalities are obvious, and so $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \mathsf{W}_3$. ◆

Our next result contains several conditions that are equivalent to the definition of a direct sum.

Theorem 5.10. *Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space* V. *The following conditions are equivalent.*

(a) $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

(b) $\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$ *and, for any vectors $v_1, v_2, \ldots, v_k$ such that $v_i \in \mathsf{W}_i$ $(1 \le i \le k)$, if $v_1 + v_2 + \cdots + v_k = 0$, then $v_i = 0$ for all i.*

(c) *Each vector $v \in \mathsf{V}$ can be uniquely written as $v = v_1 + v_2 + \cdots + v_k$, where $v_i \in \mathsf{W}_i$.*

(d) *If γ_i is an ordered basis for W_i $(1 \le i \le k)$, then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for* V.

(e) *For each $i = 1, 2, \ldots, k$, there exists an ordered basis γ_i for W_i such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for* V.

Proof. Assume (a). We prove (b). Clearly

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i.$$

Now suppose that $v_1, v_2, \ldots, v_k$ are vectors such that $v_i \in \mathsf{W}_i$ for all i and $v_1 + v_2 + \cdots + v_k = 0$. Then for any j

that this represen... where $w_i \in \mathsf{W}_i$ for all i. Then

$$(v_1 - w_1) + (v_2 - w$$

But $v_i - w_i \in \mathsf{W}_i$ for all i, and the... $v_i = w_i$ for all i, proving the uniqu...

Now assume (c). We prove (d)... W_i. Since

V

by (c), it follows that $\gamma_1 \cup \gamma_2 \cup$ ... set is linearly independent, cons... $i = 1, 2, \ldots, k$) and scalars a_{ij} su...

For each i, set

Then for each i, $w_i \in \text{span}(\gamma_i)$

$$w_1 + w_2 +$$

Since $0 \in \mathsf{W}_i$ for each i and 0 ... that $w_i = 0$ for all i. Thus