Math 2331 Homework Solutions

1. A $n\times n$, not invertible. By the Invertible Matrix Theorem,

A is invertible if and only if the columns of A are linearly independent.

8 Since A is not invertible, the columns of A are not linearly independent.

This means that the columns of A are linearly dependent.

2. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution $[A \; I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

8 $\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$ $\quad A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

Check $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

3. Invertible? a. $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 7 & 15 \\ 3 & 10 & 23 \end{bmatrix}$ $\quad [A \; I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 2 & 7 & 15 & 0 & 1 & 0 \\ 3 & 10 & 23 & 0 & 0 & 1 \end{array}\right]$

8 $\to \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 8 & 7 & -7 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & 1 & -4 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$ $\quad A^{-1} = \begin{bmatrix} 11 & 1 & -4 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 7 & 15 \\ 3 & 10 & 22 \end{bmatrix} \to \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 2 & 7 & 15 & 0 & 1 & 0 \\ 3 & 10 & 22 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 \end{array}\right]$

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$\to \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$ B is not invertible because B has only 2 pivots.

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4. Find a basis for Col(A), and a basis for Nul(A)

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -6 & 4 \\ 4 & 1 & 5 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -6 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 2 & -6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$\quad P \; P \; F \; F$

Pivot cols of A = basis for $Col(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Special solutions to $AX=0$: x_3, x_4 are free variables

$x_3 = 1 \Rightarrow x_1 = -2,\ x_2 = 3,\ \underline{x_4 = 0}$; $X = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$. Bases for Nul(A) is $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$x_3 = 0, x_4 = 1 \to x_1 = -3,\ x_2 = -2$ $\quad Y = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

5. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ $x = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. Find $[X]_{\mathcal{B}}$ = coord. vector

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Solution Solve $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 \\ 3 & 2 & 5 \end{bmatrix}$ Augmented

$$\to \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -11/7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 19/7 \\ 0 & 1 & -11/7 \end{bmatrix}$$

$$[X]_{\mathcal{B}} = \begin{bmatrix} 19/7 \\ -11/7 \end{bmatrix}$$

6. A is 2×5, $Ax = b$ has a solution for every $b \in \mathbb{R}^3$

(6) Then rank $A = 2$, $\dim(Nul\,A) = 5 - 2 = 3$

(6) 7. If A is 12×8, $\dim Nul(A) \le$ # cols $= 8$. $\dim Nul A = 8$ if $A = 0_{12 \times 8}$.

8. Find A^{-1}, $A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ $\quad [A \; I] = \begin{bmatrix} 2 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1 \end{bmatrix}$

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$$\to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$$