EXERCISES FOR MATH 2331 DUE APRIL 15

(1) Find an orthogonal basis for the column space of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \qquad (8pts)$$

Solution Use Gram-Schmidt: Let $A = [\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3]$. Let $\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Let $\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{v}_1^T \mathbf{a}_2}{\|\mathbf{v}_1\|^2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

Let $\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{v}_1^T \mathbf{a}_3}{\|\mathbf{v}_1\|^2} - \frac{\mathbf{v}_2^T \mathbf{a}_3}{\|\mathbf{v}_2\|^2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 0\mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

Then $[\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3]$ is an orthogonal basis for $Col A$.

(2) Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ by (a) constructing the normal equations for $\hat{\mathbf{x}}$ and (b) solving for $\hat{\mathbf{x}}$. Then (c) find the distance from $A\hat{\mathbf{x}}$ to $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solution

(a) Normal equations: $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$: $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$. $A^T\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Normal equations are:

$$\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \qquad (6\text{ pts})$$

(b) Solve: $\hat{\mathbf{x}} = \left(A^T A\right)^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. (6 pts)

(c) $\|A\hat{x} - \mathbf{b}\|$: $A\hat{x} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$. $A\hat{x} - \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ and $\|A\hat{x} - \mathbf{b}\| = 1$. (6 pts)

(3) Find the equation $y = ax + b$ of the least-squares line that best fits these data points:

$$(-2, 4)\ \ (-1, 2)\,,\ \ (1, 0)\,,\ \ (2, 0)\,.$$

Solution

(a) Write the equations that would be true if the line $y = ax + b$ passed through all four points:

$$\begin{aligned} -2a + b &= 4 \\ -1a + b &= 2 \\ 1a + b &= 0 \\ 2a + b &= 0 \end{aligned} \qquad \text{(4 pts)}$$

(b) Next, write this system in matrix form:

$$\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \text{ or } A\mathbf{x} = \mathbf{c}. \qquad \text{(4 pts)}$$

(c) Next, multiply both sides of this equation by A^T.

$$A^T A = \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}, \quad A^T \mathbf{c} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}.$$

So the new system is

$$\begin{bmatrix} -10 \\ 6 \end{bmatrix} \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}. \qquad \text{(4 pts)}$$

(d) The solution is

$$\hat{x_1} = -1 = a, \quad \hat{x_2} = \frac{3}{2} = b.$$

The least-squares line for these data points is

$$y = -x + \frac{3}{2}. \qquad \text{(4 pts)}$$

An alternate method of solution is available since the columns of A, $\mathbf{a}_1,\ \mathbf{a}_2$ are orthogonal. See p. 366 of text. The orthogonal projection of $\mathbf{c}$ onto $ColA$ is

$$\hat{\mathbf{c}} = \frac{\mathbf{a}_1^T \mathbf{c}}{\|\mathbf{a}_1\|^2}\mathbf{a}_1 + \frac{\mathbf{a}_2^T \mathbf{c}}{\|\mathbf{a}_2\|^2}\mathbf{a}_2. \qquad \text{(8pts)}$$

Then $\hat{x_1} = \frac{\mathbf{a}_1^T \mathbf{c}}{\|\mathbf{a}_1\|^2} = \frac{-10}{10} = -1$, $\hat{x_2} = \frac{\mathbf{a}_2^T \mathbf{c}}{\|\mathbf{a}_2\|^2} = \frac{6}{4} = \frac{3}{2}$ (8pts)

(4) Let

$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix},\ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Let $W = Span\,[\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3]$. Express $\mathbf{v}$ as the sum of a vector in W and a vector orthogonal to W.

Solution Since $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthogonal set, we can compute

$$\begin{aligned} proj_W\mathbf{v} &= \frac{\mathbf{u}_1^T\mathbf{v}}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{u}_2^T\mathbf{v}}{\|\mathbf{u}_2\|^2}\mathbf{u}_1 + \frac{\mathbf{u}_3^T\mathbf{v}}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 \\ &= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{-5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} \qquad \text{(8pts)} \end{aligned}$$

Then $\mathbf{v}^\perp = \mathbf{v} - proj_W\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$. Check that $\mathbf{v}^\perp \perp W$. (4 pts)