Math 2331 Homework Solutions

1. $v_1 = \begin{bmatrix}2\\0\\3\end{bmatrix}$ $v_2 = \begin{bmatrix}1\\3\\0\end{bmatrix}$ $v_3 = \begin{bmatrix}1\\4\\1\end{bmatrix}$. $A = [v_1\ v_2\ v_3] = \begin{bmatrix}2&1&1\\0&3&4\\3&0&1\end{bmatrix}$

(6)

A is row equivalent to $\begin{bmatrix}1&1/2&1/2\\0&3&4\\0&-3/2&-1/2\end{bmatrix} \sim \begin{bmatrix}1&1/2&1/2\\0&1&4/3\\0&1&1/3\end{bmatrix} \to \begin{bmatrix}1&1/2&1/2\\0&1&4/3\\0&0&-1\end{bmatrix}$

A has 3 pivot positions and $Ax=0$ has no free variables

So the column vectors of A, $\{v_1, v_2, v_3\}$, are linearly independent

2. $v_1 = \begin{bmatrix}2\\0\\3\end{bmatrix}$ $v_2 = \begin{bmatrix}1\\3\\0\end{bmatrix}$. $\{v_1, v_2\}$ is a subset of a linearly independent set. So $\{v_1, v_2\}$ is linearly independent

(4)

Alt. $A = [v_1, v_2]$ has 2 pivots + no free variables

3. $v_1 = \begin{bmatrix}2\\3\end{bmatrix}$ $v_2 = \begin{bmatrix}1\\-5\end{bmatrix}$, $v_3 = \begin{bmatrix}3\\1\end{bmatrix}$

(4)

3 vectors in $\mathbb{R}^2$ must be linearly dependent.

Alternate: $A = [v_1, v_2, v_3]$ has 2 pivots + 1 free variable.

4. $v_1 = \begin{bmatrix}3\\6\\1\end{bmatrix}$ $v_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $v_3 = \begin{bmatrix}9\\8\\8\end{bmatrix}$.

(4)

Any set that contains the 0 vector is linearly dependent.

5. $T\Big(\overset{u}{\begin{bmatrix}2\\3\end{bmatrix}}\Big) = \begin{bmatrix}1\\4\end{bmatrix}$, $T\Big(\overset{v}{\begin{bmatrix}5\\-2\end{bmatrix}}\Big) = \begin{bmatrix}5\\-1\end{bmatrix}$, T is linear

(9)

(3) a. Then $T(3u) = 3T(u) = 3T\begin{pmatrix}2\\3\end{pmatrix} = 3\cdot\begin{pmatrix}1\\4\end{pmatrix} = \begin{bmatrix}3\\12\end{bmatrix}$

(3) b. $T(u-2v) = T(u) - 2T(v) = \begin{bmatrix}1\\4\end{bmatrix} - 2\begin{bmatrix}5\\-1\end{bmatrix} = \begin{bmatrix}-9\\6\end{bmatrix}$

(3) c. $T(2u+v) = 2T(u) + T(v) = 2\cdot\begin{bmatrix}1\\4\end{bmatrix} + \begin{bmatrix}5\\-1\end{bmatrix} = \begin{bmatrix}7\\7\end{bmatrix}$

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6. A has reduced echelon form $R = \begin{bmatrix} 1 & 5 & 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$

a. Describe the solution set (to $Ax=0$).

(8)

Solution. $A\begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = \vec{0}$ iff
$$\begin{aligned} x_1 + 5x_2 &= 0 \\ x_3 + 3x_4 + 2x_6 &= 0 \\ x_5 - 3x_6 &= 0 \end{aligned}$$

Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \\ -3x_4 - 2x_6 \\ x_4 \\ 3x_6 \\ x_6 \end{bmatrix} = x_2\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

b. Find the solution with $x_2 = 1$, $x_4 = 0$, $x_6 = 0$

(2)

Solution $\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

c. Suppose the first column of A is $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.
What is the second column?

(6)

Solution Let A have columns $[a_1 \cdots a_6]$

Then $A\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -5a_1 + a_2 = \vec{0}$. So $a_2 = 5a_1 = \begin{bmatrix} 10 \\ -5 \\ 20 \end{bmatrix}$

d. Find the solution with $x_2 = 0$, $x_4 = 1$, $x_6 = 0$

(2)

Solution $\begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

e. If, in addition to the above, $\vec{a}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, find a_4

(6)

Solution Now $[a_1, \ldots, a_6]\begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -3a_3 + a_4 = 0$

So $a_4 = 3a_3 = \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix}$

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6 F Are the pivot columns of R linearly independent or dependent?

Solution The pivot columns of any matrix are linearly independent

(6) The pivot columns of R are $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ which are clearly independent

Because $RX=0$ if and only if $AX=0$,
the columns of R have the same dependence relations as the columns of A. The pivot columns of R have no dependence relations, so neither do the pivot columns of A.

7. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ (be a linear transformation) and suppose $\{v_1, \dots, v_p\}$ is a set of vectors such that $\{T(v_1), \dots, T(v_p)\}$ is linearly independent. Show that $\{v_1, \dots, v_p\}$ is linearly independent

Proof One method: Prove the contrapositive. That is, show

(8) that if $\{v_1, \dots, v_p\}$ is linearly dependent then $\{T(v_1), \dots, T(v_p)\}$ is linearly dependent.

If $\{v_1, \dots, v_p\}$ is linearly dependent, then there are scalars $c_1, \dots, c_p$, not all zero, such that

$$c_1 v_1 + \dots + c_p v_p = 0$$

$$\text{Then } 0 = T(0) = T(c_1 v_1 + \dots + c_p v_p) = c_1 T(v_1) + \dots + c_p T(v_p)$$

Since $c_1, \dots, c_p$ are not all zero, the vectors $\{T(v_1), \dots, T(v_p)\}$ are linearly dependent.

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8 $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear & first rotates points counterclockwise $\pi/3$. then performs a horizontal shear that maps e_2 to e_2+3e_1 and leaves e_1 unchanged. Find the standard matrix.

Method 1. $T(e_1) = S(R(e_1))$ S = shear, R = rotation

$= S\left(\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}\right) = S\left(\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix}\right) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \sqrt{3}/2\,(e_2+3e_1)$

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$= \begin{pmatrix} 1/2 + 3\sqrt{3}/2 \\ \sqrt{3}/2 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = S\begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} = S\left(-\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2\right) = -\frac{\sqrt{3}}{2}e_1 + 1/2(e_2+3e_1)$

$= \begin{pmatrix} \frac{(3-\sqrt{3})}{2} \\ 1/2 \end{pmatrix}$

Then T has standard matrix $\begin{pmatrix} 1/2 + 3\sqrt{3}/2 & \frac{3-\sqrt{3}}{2} \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

Method 2 R has standard matrix $\begin{pmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

S has standard matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Then $T = S \circ R$ has standard matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

$= \begin{pmatrix} 1/2 + 3\sqrt{3}/2 & \frac{(3-\sqrt{3})}{2} \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

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9. If the columns of AB are independent

$AB = [Ab_1 \cdots Ab_p]$ so $\{Ab_1, \ldots, Ab_p\}$ is linearly independent

then $\{b_1, \ldots, b_p\}$ must be linearly indep by problem 7.

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