EXERCISES FOR MATH 2331 DUE APRIL 1

(1) Find the characteristic polynomial and the (real) eigenvalues for the matrix

$$A = \begin{bmatrix} -3 & 3 \\ 8 & -1 \end{bmatrix}$$

For each eigenvalue, find a basis for the corresponding eigenspace.

Solution

$$\det(A - \lambda I) = (-3-\lambda)(-1-\lambda) - 24 = \lambda^2 + 4\lambda - 21$$

This factors as $(\lambda+7)(\lambda-3)$, The eigenvalues of A are $\lambda = -7,\ 3$. With $\lambda = 3$, $(A - 3I) = \begin{bmatrix} -6 & 3 \\ 8 & -4 \end{bmatrix}$. The null space of $A - 3I$ is the eigenspace for $\lambda = 3$. A basis for this eigenspace is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. $A + 7I = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$. A basis for the eigenspace for $\lambda = -7$ is $\left\{ \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$.

(2) The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$ has eigenvalues $\lambda_1 = -8$ and $\lambda_2 = 2$. Find eigenvectors for A that correspond to these eigenvalues.

Solution $A + 8I = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}$ An eigenvector for $\lambda = -8$ is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.
$A - 2I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$ An eigenvector for $\lambda = 2$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(3) Find an invertible matrix P and a real matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} = PCP^{-1}.$$

(See Theorem 9 on page 301).

Solution

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(5-\lambda) + 8 = \lambda^2 - 6\lambda + 13 \\ &= (\lambda - 3)^2 + 4. \end{aligned}$$

The eigenvalues are $\lambda = 3 \pm 2i$. $A - (3-2i)I = \begin{bmatrix} -2+2i & 2 \\ -4 & 2+2i \end{bmatrix}$.
An eigenvector for $\lambda = 3 - 2i$ is $\begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Let $P = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ as prescribed in Theorem 9. Then P is invertible and $P^{-1}AP = C = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$, so that $A = PCP^{-1}$.

(4) Find an invertible matrix P and a diagonal matrix D such that

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} = PDP^{-1}.$$

Solution $\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$. A has eigenvalues 2, 4.
$A - 2I = \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}$. An eigenvector for $\lambda = 2$ is $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
$A - 4I = \begin{bmatrix} -3 & -3 \\ 1 & 3 \end{bmatrix}$. An eigenvector for $\lambda = 4$ is $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
Then $P = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(5) If $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$, find all of the eigenvalues of A , their algebraic multiplicities, and a basis for each eigenspace. Is A diagonalizable?
Solution Because A is upper triangular, the diagonal entries are the eigenvalues: $\lambda_1 = 2$ with algebraic multiplicity 3, and $\lambda_2 = 3$ with algebraic multiplicity 2.

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Both of these matrices have rank 4, so the dimension of each eigenspace is 1.

A basis for the eigenspace for $\lambda = 2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, and for $\lambda = 3$, $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.