EXERCISES FOR MATH 2331 DUE APRIL 8
WITH SOLUTIONS

(1) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ \frac{1}{2} & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & -2 \end{bmatrix} = PDP^{-1}.$$

Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

Solution The eigenvalues of A are the diagonal entries of D: 2, 2, 1. The eigenvectors are the columns of P, in the same order as the eigenvalues in $D : \lambda = 2 : \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$,
$\lambda = 1 : \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(2) Diagonalize $A = \begin{bmatrix} -4 & -1 \\ 1 & -2 \end{bmatrix}$ or explain why it is not diagonalizable.

Solution $\det(A - \lambda I) = \lambda^2 + 6\lambda + 9 = (\lambda - 3)^2$. $\lambda = -3$ is an eigenvalue with algebraic multiplicity 2.

$A + 3I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ has rank 1, so that the eigenspace for $\lambda = -3$ has dimension 1. Thus A is not diagonalizable.

(3) Diagonalize $A = \begin{bmatrix} -4 & 2 \\ 1 & -2 \end{bmatrix}$ or explain why it is not diagonalizable.

Solution $\det(A - \lambda I) = \lambda^2 + 6\lambda + 6$. The roots are $\lambda = -3 \pm \sqrt{9-6} = -3 \pm \sqrt{3}$. A is 2×2 with 2 distinct eigenvalues, so A is diagonalizable. With $\lambda = -3 + \sqrt{3}$, $A - (-3 + \sqrt{3})I = \begin{bmatrix} -1 - \sqrt{3} & 2 \\ 1 & 1 - \sqrt{3} \end{bmatrix}$. A vector in the null space of this is $\mathbf{u} = \begin{bmatrix} -1 + \sqrt{3} \\ 1 \end{bmatrix}$. An eigenvector for $\lambda = -3 - \sqrt{3}$ is $\mathbf{v} = \begin{bmatrix} -1 - \sqrt{3} \\ 1 \end{bmatrix}$. Then with $P = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$ and $D = \begin{bmatrix} -1 + \sqrt{3} & 0 \\ 0 & -1 - \sqrt{3} \end{bmatrix}$, $A = PDP^{-1}$.

(4) Suppose $\mathbf{v},\ \mathbf{u},\ \mathbf{w}$ are vectors in $\mathbb{R}^n$ such that $\|\mathbf{v}\| = 4$, $\|\mathbf{u}\| = 3$, $\|\mathbf{w}\| = 6$, $\mathbf{v}\cdot\mathbf{u} = 10$, $\mathbf{v}\cdot\mathbf{w} = 7$, $\mathbf{u}\cdot\mathbf{w} = -2$.

(a) Find $\|\mathbf{v}+\mathbf{u}\|$.

Solution

$$\begin{aligned}\|\mathbf{v}+\mathbf{u}\|^2 &= (\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{u}\\ &= \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{u} + \|\mathbf{u}\|^2 = 16 + 2\cdot 7 + 9 = 39.\end{aligned}$$

(b) Find the projection of $\mathbf{w}$ onto the span of $\mathbf{u}$.

Solution $Proj_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u}^T\mathbf{w}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} = \frac{-2}{9}\mathbf{u}$. Since I did not provide $\mathbf{u}$, this is as far as we can go.

(5) Let $\mathbf{v}_1 = \begin{bmatrix}3\\4\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}-4\\3\end{bmatrix}$, $\mathbf{u} = \begin{bmatrix}1\\6\end{bmatrix}$. Verify that $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is an orthogonal basis for $\mathbb{R}^2$ and express $\mathbf{u}$ as a linear combination of $\{\mathbf{v}_1,\ \mathbf{v}_2\}$.

Solution
$\mathbf{v}_1\cdot\mathbf{v}_2 = 3(-4)+4\cdot 3 = 0$, so $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is an orthogonal set of 2 nonzero vectors in $\mathbb{R}^2$. By Theorem 4, the set is linearly independent, and the number of vectors equals the dimension of the space, so the set is a basis for $\mathbb{R}^2$. Then by Theorem 5, $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where $c_i = \frac{\mathbf{v}_i^T\mathbf{u}}{\mathbf{v}_i^T\mathbf{v}_i}$, so $c_1 = \frac{27}{25}$, $c_2 = \frac{14}{25}$.