Proof of Schur's Theorem

David H. Wagner

In this note, I provide more detail for the proof of Schur's Theorem found in Strang's Introduction to Linear Algebra [1].

**Theorem 0.1.** If $A$ is a square real matrix with real eigenvalues, then there is an orthogonal matrix $Q$ and an upper triangular matrix $T$ such that $A = QTQ^T$.

*Proof.* Note that $A = QTQ^T$ $\iff$ $AQ = QT$. Let $q_1$ be an eigenvector of norm 1, with eigenvalue $\lambda_1$. Let $q_2, \ldots, q_n$ be any orthonormal vectors orthogonal to $q_1$. Let $Q_1 = [q_1, \ldots, q_n]$. Then $Q_1^T Q_1 = I$, and

$$Q_1^T A Q_1 = \begin{pmatrix} \lambda_1 & \cdots \\ 0 & A_2 \end{pmatrix}$$

Now I claim that $A_2$ has eigenvalues $\lambda_2, \ldots, \lambda_n$. This is true because

$$\det (A - \lambda I) = \det Q_1^T (A - \lambda I) Q_1 = \det (Q_1^T (A - \lambda I) Q_1)^T$$

$$= \det (Q_1^T A Q_1 - \lambda Q_1^T Q_1) = \det \begin{pmatrix} (\lambda_1 - \lambda) & \cdots \\ 0 & (A_2 - \lambda I) \end{pmatrix}$$

$$= (\lambda_1 - \lambda) \det (A_2 - \lambda I).$$

So $A_2$ has real eigenvalues, namely $\lambda_2, \ldots, \lambda_n$. Now we proceed by induction. Suppose we have proved the theorem for $n = k$. Then we use this fact to prove the theorem is true for $n = k + 1$. Note that the theorem is trivial if $n = 1$.

So for $n = k + 1$, we proceed as above and then apply the known theorem to $A_2$, which is $k \times k$. We find that $A_2 = Q_2 T_2 Q_2^T$. Now this is the hard part. Let $Q_1$ and $A_2$ be as above, and let

$$Q = Q_1 \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$ 

Then

$$AQ = A Q_1 \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix} = Q_1 \begin{pmatrix} \lambda_1 & \cdots \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

$$= Q_1 \begin{pmatrix} \lambda_1 & \cdots \\ 0 & A_2 Q_2 \end{pmatrix} = Q_1 \begin{pmatrix} \lambda_1 & \cdots \\ 0 & Q_2 T_2 \end{pmatrix}$$

$$= Q_1 \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots \\ 0 & T_2 \end{pmatrix} = QT,$$

where $T$ is upper triangular. So $AQ = QT$, or $A = QTQ^T$. $\square$

That's all, folks!
References