Math 2433 Exam 2 Review Solutions

1.a. Find the domain and range:

i. $f(x,y) = \ln(\sqrt{1+x^2+y^2})$

domain $= \{(x,y): -\infty < x < \infty, -\infty < y < \infty\}$
$= \mathbb{R}^2$.

range: Since $\sqrt{1+x^2+y^2} \geq 1$

$\ln(\sqrt{1+x^2+y^2}) \geq \ln(1) = 0$.

Also $1+x^2+y^2 \to \infty$ as $x \to \infty$ or $y \to \infty$

so Range $(\ln(\sqrt{1+x^2+y^2})) = \{z: 0 \leq z < \infty\}$.

ii. $F(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2-1}}$

Domain $= \{(x,y,z): x^2+y^2+z^2 > 1\}$

Range. $F(x,y,z) > 0$

As $x^2+y^2+z^2 \to 1$ $f(x,y,z) \to \infty$.

As $x^2+y^2+z^2 \to \infty$ $F(x,y,z) \to 0$.

Since domain of F is connected,
Intermediate Value Theorem $\Rightarrow$
range of F is an interval.
So range of $F = \{u: 0 < u < \infty\}$.

b. Identify the level curves/surfaces of each of the following functions.

(i) $f(x,y) = e^{-4x^2-y^2}$

If $f(x,y) = c$ then $-4x^2-y^2 = \ln c = d$

level curves are $\{(x,y): 4x^2+y^2 = d\}$

and are ellipses.

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1b ii level surfaces of $f(x,y,z) = 2x+3y+6z$

if $f(x,y,z) = c$ then

$$2x+3y+6z = c$$

Level surfaces are planes with normal vector $2i+3j+6k$

1C. An open rectangular container (i.e., no top) is to have a volume of 12 ft³. The cost of the material for the sides is \$3 per square foot and the cost for the base is \$5 per sq. ft. Express the total cost C of the container as a function of its length x and width y.

Solution $xyz = 12$ so $z = \frac{12}{xy}$.

$$\text{cost} = (2xz + 2yz)\cdot \$3 + xy \cdot \$5.$$

$$= \frac{2(x+y)\cdot 12}{xy}\cdot \$3 + xy\cdot \$5.$$

$$= \left(\frac{72}{y} + \frac{72}{x} + 5xy\right)(\$).$$

2. Let $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$

a. Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ if $(x,y)\to(0,0)$ along the x-axis.

Soln $= \lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} \dfrac{0}{x^4} = 0.$

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2. $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$.

b. Find $\lim_{(x,y)\to(0,0)}$ if $(x,y)\to(0,0)$ along the y-axis

Soln $= \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \dfrac{0}{y^2} = 0.$

c. Find $\lim_{(x,y)\to(0,0)}$ if $(x,y)\to(0,0)$ along
The line $y = mx$.

Soln. $= \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \dfrac{2x^2mx}{x^4+(mx)^2}$

$= \lim_{x\to 0} \dfrac{2mx^3}{x^4+m^2x^2} = \lim_{x\to 0} \dfrac{2mx}{x^2+m^2} = 0.$

d. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ if $(x,y)\to(0,0)$

along the parabola $y = \lambda x^2$ $\lambda > 0$.

Soln. $= \lim_{x\to 0} f(x,\lambda x^2) = \lim_{x\to 0} \dfrac{2x^2(\lambda x^2)}{x^4+\lambda^2 x^4}$

$= \lim_{x\to 0} \dfrac{2\lambda x^4}{x^4(1+\lambda^2)} = \lim_{x\to 0} \dfrac{2\lambda}{1+\lambda^2} = \dfrac{2\lambda}{1+\lambda^2}.$

e. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? No.

3. Let $f(x,y) = y^2e^{xy} + \frac{x}{y}$. Calculate f_{xx} and f_{yx}.

Soln $f_x = y^2e^{xy}\cdot y + \frac{1}{y} = y^3e^{xy} + \frac{1}{y}$

$f_y = 2ye^{xy} + y^2xe^{xy} - \frac{x}{y^2}$

$= (2y + xy^2)e^{xy} - \frac{x}{y^2}.$

$f_{xx} = y^4e^{xy}$

$f_{xy} = 3y^2e^{xy} + y^3xe^{xy} - \frac{1}{y^2}.$

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3b. Let $z = \ln(\sqrt{x^2+y^2})$. Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$. (Note $z = \frac{1}{2}\ln(x^2+y^2)$)

Soln. $\frac{\partial z}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x \qquad \frac{\partial z}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y$

So $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2+y^2}{x^2+y^2} = 1.$

c. Let $u = x^2 - 2y^2 + z^3$ where $x = \sin t$, $y = e^{2t}$, $z = 3t$. Calculate $\frac{du}{dt}$ and express your answer in terms of t.

Soln.

One way $\quad u = (\sin t)^2 - 2(e^{2t})^2 + (3t)^3$

$\frac{du}{dt} = 2\sin t\cos t - 4e^{2t}\cdot 2e^{2t} + 3(3t)^2\cdot 3$

Chain rule way $\quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$

$= 2x\cdot\cos t + (-4y)\cdot 2e^{2t} + 3z^2\cdot 3$

$= 2\sin t\cos t - 8e^{4t} + 9(3t)^2$

d. Let $z = e^{2x}\ln y$ where $x = u^2 - 2v$ and $y = v^2 - 2u$. Calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Solution $\quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

$= 2e^{2x}\ln y\cdot 2u + \frac{e^{2x}}{y}\cdot(-2)$

$= 2e^{2(u^2-2v)}\ln(v^2-2u)\cdot 2u - \frac{2e^{2(u^2-2v)}}{v^2-2u}$

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4. Let $f(x,y) = x\tan^{-1}(\frac{y}{x})$ and
$F(x,y,z) = x^2 + 3yz + 4xy$.

a i. Find the gradient of F.

Soln $\nabla F = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$.
$= (2x+4y)i + (3z+4x)j + 3yk$.

ii Determine the direction in which f decreases most rapidly at the point $(2,2)$. At what rate is f decreasing (in this direction)?

Soln f decreases most rapidly in the direction of $-\nabla f(2,2)$

$$\nabla f(x,y) = \left(\tan^{-1}(y/x) + x\frac{1}{1+(y/x)^2}\cdot(-y/x^2)\right)i + x\frac{1}{1+(\frac{y}{x})^2}\frac{1}{x}j$$

$$\nabla f(2,2) = \left(\tan^{-1}(1) + 2\frac{(-2)}{2^2+2^2}\right)i + \frac{1}{1+1}j$$

$$= (\pi/4 - \tfrac{1}{2})i + \tfrac{1}{2}j.$$

$$\|\nabla f(2,2)\| = \sqrt{(\pi/4 - 1/2)^2 + (1/2)^2} = \sqrt{\pi^2/16 - \pi/4 + \tfrac{1}{4} + \tfrac{1}{4}}$$
$$= \sqrt{\pi^2/16 - \pi/4 + \tfrac{1}{2}}$$

A unit vector in the direction of $-\nabla f(2,2)$ is

$$u = \frac{(1/2 - \pi/4)i - 1/2\,j}{\sqrt{\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}}}$$

The rate at which f decreases in this direction is $\|\nabla f(2,2)\| = \sqrt{\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}}$

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4b. Find the directional derivative of F at the point $(1,1,-5)$ in the direction of the vector $a = 2i + 3j - \sqrt{3}k$.

Solution.

$$F_u'(1,1,-5) = \nabla F(1,1,-5) \cdot u_a$$

$$u_a = \frac{(2i + 3j - \sqrt{3}k)}{\sqrt{4+9+3}} = \frac{(2i+3j-\sqrt{3}k)}{4}$$

$$\nabla F(1,1,-5) = (2x+4y)i + (3z+4x)j + (3y)k$$
$$= (2+4)i + (-15+4)j + 3k$$
$$= 6i - 11j + 3k$$

$$\nabla F(1,1,-5) \cdot u_a = (6i - 11j + 3k) \cdot \frac{(2i+3j-\sqrt{3}k)}{4}$$

$$= \frac{12 - 33 - 3\sqrt{3}}{4} = \frac{-21 - 3\sqrt{3}}{4}.$$

4c Find an equation for the tangent plane to the level surface $F(x,y,z) = 3$ at the point $(3,-1,-2)$.

Soln 1st Check $F(3,-1,-2) = 3$?

$$= 3^2 + 3\cdot(-1)(-2) + 4(\cdot 3)(-1)$$
$$= 9 + 6 - 12 = 3 \checkmark$$

A normal vector to tangent plane is $\nabla F(3,-1,-2)$

$$= (2x+4y)i + (3z+4x)j + 3yk$$
$$= (6-4)i + (-6+12)j + (-3)k$$
$$= 2i + 6j - 3k.$$

Point $= (3,-1,-2)$. so $2(x-3) + 6(y+1) - 3(z+2) = 0.$

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4a. Find an equation for the tangent plane and scalar parametric equations for the normal line to the surface $z = f(x,y)$ at the point $(2,-2,-\pi/2)$.

Soln. Tangent plane.

$$z - f(2,-2) = \frac{\partial f}{\partial x}(2,-2)(x-2) + \frac{\partial f}{\partial y}(2,-2)(y+2)$$

[Check $f(2,-2) = -\pi/2$?

$$f(2,-2) = 2\tan^{-1}\left(\frac{y}{x}\right) = 2\tan^{-1}(-1) = -\pi/2.] \checkmark$$

So eqn of tang plane is

$$z - (-\pi/2) = \left(\tan^{-1}\left(\frac{-2}{2}\right) + 2\frac{1}{1+\left(\frac{2}{2}\right)^2}\left(\frac{2}{2^2}\right)\right)(x-2)$$

$$+ \left(2\frac{1}{1+\left(\frac{-2}{2}\right)^2}\cdot\frac{1}{2}\right)(y+2), \text{ or}$$

$$z + \pi/2 = \left(-\pi/4 + \frac{1}{2}\right)(x-2) + \frac{1}{2}(y+2)$$

Normal line has direction vector $\nabla G(2,-2,-\pi/2)$ where $G(x,y,z) = f(x,y) - z$

So $\nabla G = f_x i + f_y j - k$

$$= \left(\frac{1}{2} - \pi/4\right)i + \frac{1}{2}j - k.$$

So the normal line has vector parametric equation

$$r(t) = (2,-2,-\pi/2) + t\left(\left(\frac{1}{2} - \frac{\pi}{4}\right), 1/2, -1\right)$$

and scalar parametric equations

$$x = 2 + \left(\frac{1}{2} - \pi/4\right)t$$

$$y = -2 + 1/2\, t$$

$$z = -\pi/2 - t.$$

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5. Determine whether F is the gradient of a function f. If it is, find all such f.

a. $F(x,y) = (3x^2y^2 + 3y + x)i + (2x^3y + 3xy - \sqrt{y})j$

Soln If $F = \nabla f$ then

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 3y + x \qquad \frac{\partial f}{\partial y} = 2x^3y + 3xy - \sqrt{y}.$$

Then $\frac{\partial^2 f}{\partial y \partial x} = 6x^2y + 3 \;=\; \frac{\partial^2 f}{\partial x \partial y} = 6x^2y + 3y$

Since these are not equal, F is not the gradient of a function f.

b. $F(x,y) = (2xe^y + 4xy + e^{2x})i + (x^2e^y + 2x^2 + \cos y - 1)j$

Soln As in a, we need

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(2xe^y + 4xy + e^{2x}) = \frac{\partial^2 f}{\partial x \partial y}$$

$$= \frac{\partial}{\partial x}(x^2e^y + 2x^2 + \cos y - 1)$$

1st $= 2xe^y + 4x$ $\qquad$ 2nd $= 2xe^y + 4x$.

Since, in addition, F is continuously differentiable on $\mathbb{R}^2$, by a theorem of Section 15.4, there is a function $f(x,y)$ such that $\nabla f(x,y) = F(x,y)$. $f(x,y)$ can be found as follows.

$\frac{\partial f}{\partial x} = 2xe^y + 4xy + e^{2x}$ $\qquad f(x,y) = \int 2xe^y + 4xy + e^{2x}\,dx$ (treat y as constant)

$$= x^2e^y + 2x^2y + \frac{e^{2x}}{2} + \underbrace{g(y)}_{\text{constant of integration}}.$$

Then $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2e^y + 2x^2y + \frac{e^{2x}}{2} + g(y)\right)$

$$= x^2e^y + 2x^2 + g'(y) = x^2e^y + 2x^2 + \cos y - 1$$

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5b cont. So $x^2e^y + 2x^2 + g'(y) = x^2e^y + 2x^2 + \cos y - 1$

Thus $g'(y) = \cos y - 1$.

Note that the right side must depend on y alone! It not, you made a mistake!

Solving $g(y) = \int \cos y - 1 \, dy = \sin y - y + c$.

So $f(x,y) = x^2e^y + 2x^2y + \frac{e^{2x}}{2} + \sin y - y + c$.

These are all of the solutions (Note the "$+c$").

Problem 6 a Find the stationary points of $x^2 + 2y^2 - x^2y = f(x,y)$

Soln $\nabla f(x,y) = (2x - 2xy, 4y - x^2)$

$= (0,0)$ if

$$\left.\begin{array}{l} 2x - 2xy = 0 \\ -x^2 + 4y = 0. \end{array}\right\} \rightarrow \begin{array}{l} 2x(1-y) = 0 \\ 4y - x^2 = 0. \end{array}$$

If $x = 0$ $y = 0$

If $x \neq 0$ $y = 1$, $x^2 = 4 \cdot 1 = 4$ $x = \pm 2$.

$(\pm 2, 1)$ $(0,0)$. are all of the stationary points of f.

b For each stationary pt P found in (a), determine whether f has a local minimum, a local maximum, or a saddle pt at P.

Soln $f_{xx} = 2 - 2y$ $f_{xy} = -2x$, $f_{yy} = 4$

At $(0,0)$ $A = f_{xx} = 2$ $B = f_{xy} = 0$ $C = f_{yy} = 4$

$AC - B^2 = 8 > 0$ $A > 0$ local min at $(0,0)$

At $(2,1)$ $A = 2 - 2 = 0$ $B = -4$ $C = 4$

$AC - B^2 = -16 < 0$ saddle pt.

Similarly $(-2,1)$ is a saddle pt.

Math 2433 Exam 2 Review Solutions.

7a Find the absolute maximum and absolute minimum value of $f(x,y) = x^2+2y^2-x$ on the closed disk $D: x^2+y^2 \le 1$.

Solution $\nabla f = (2x-1, 4y) = (0,0)$ if $x = 1/2, y=0$.

$(\frac{1}{2}, 0)$ is in D and is the only interior critical point.

Boundary stationary pts.

Method I $x=\cos\theta \quad y=\sin\theta \quad f(\cos\theta, \sin\theta)$

$= \cos^2\theta + 2\sin^2\theta - \cos\theta$

$= 1+\sin^2\theta - \cos\theta$

$\frac{d}{d\theta} f(\cos\theta, \sin\theta) = 2\sin\theta\cos\theta + \sin\theta$

$= 0$ if $\sin\theta = 0$ or $2\cos\theta + 1 = 0$

$\cos\theta = -1/2$.

$\theta = 0, \pi, \pm 2\pi/3$. $(1,0)$ $(-1,0)$, $(-1/2, \frac{\sqrt3}{2})$ $(-1/2, -\frac{\sqrt3}{2})$

Method of Lagrange $g(x,y) = x^2+y^2 = 1$

$\nabla f = \lambda\nabla g = \lambda(2x, 2y)$

$(2x-1, 4y) = \lambda(2x, 2y)$

$2x-1 = \lambda 2x$

$4y = \lambda 2y \rightarrow 2y(2-\lambda) = 0 \quad \lambda = 2$ or $y=0$.

If $y=0$ $x=\pm 1$ $(\theta = 0, \pi)$. $(1,0)$ $(-1,0)$

If $y \ne 0$ $\lambda = 2$, $2x-1 = 4x \quad -1 = 2x \quad x = -1/2$.

$\rightarrow y^2 = 1-x^2 = 3/4 \quad y = \pm\sqrt3/2$.

$(-1/2, \sqrt3/2)$, $(-1/2, -\sqrt3/2)$

Evaluate $f(1/2, 0) = \frac{1}{4} + 0 - 1/2 = -1/4$.

$f(1,0) = 1+0-1 = 0$

$f(-1,0) = 1+0+1 = 2$

$f(-1/2, \sqrt3/2) = \frac14 + 2\cdot\frac34 + 1/2 = \frac94$

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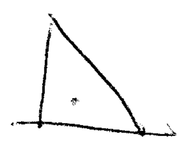
<u>7a cont.</u> $f(-1/2, -\sqrt{3}/2) = f(-1/2, \sqrt{3}/2) = 9/4$

Abs min at $(1/2, 0)$ $f(1/2, 0) = -1/4$.
Abs max at $(-1/2, \pm\sqrt{3}/2)$ $f(-1/2, \pm\sqrt{3}/2) = \frac{9}{4}$

<u>7b</u> Find the absolute maximum and absolute minimum values of $f(x,y) = 2+2x+2y-x^2-y^2$ on the closed triangular region bounded by the lines $x=0$, $y=0$, $x+y=9$.

<u>Soln</u> $\nabla f = (2-2x, 2-2y) = (0,0)$ at $(1,1)$.
Check that $(1,1)$ is in the triangle!
$(x=1>0 \quad y=1>0, \quad x+y=1+1=2<9)$
Note $f(x,y) = -(x-1)^2-(y-1)^2+4$.
So $(1,1)$ maximizes $f(x,y)$ absolutely.
The min occurs at the point on the boundary <u>farthest</u> from $(1,1)$.
This must be one of the vertices.



$f(0,0) = 2 \quad f(3,0) = 2+6+0-9-0$
$= 8-9 = -1$

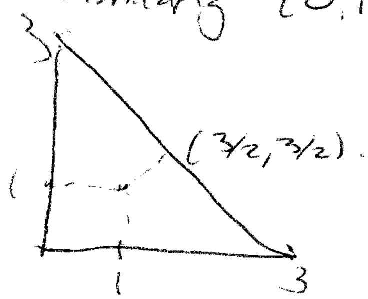
$f(0,3) = 2+0+6-0-9 = -1$.
So the max of f occurs at $(3,0)$ and $(0,3)$ with value -1.
<u>Note</u> if we use the method of Lagrange we look for (x,y) on the hypotenuse s.t.

$\nabla f = \lambda \nabla g(x,y) = (1,1) \qquad (2-2x, 2-2y) = \lambda(1,1)$
$\Rightarrow x = y = \frac{3}{2} \qquad (3/2, 3/2)$

and we look for $(x,0)$ s.t. $\nabla f = \lambda\nabla(y) = (0,1)$

Math 2433 Exam 2 Homework Solutions

7b cont. $(2-2x, 2-2y) = \lambda(0,1) \Rightarrow x = 1$,
So $(1,1)$ is stationary
Similarly $(0,1)$ is stationary on $x=0$.



But none of these points minimizes (or maximizes) f.

8a According to U.S. Postal Service regulations, the length plus the girth (perimeter of a cross-section) of a package cannot exceed 108 inches. What are the dimensions of the rectangular box of maximum volume that is acceptable for mailing? What is the maximum volume?

Solution The "length" is the longest dimension. The "girth" is the perimeter perpendicular to the length.

So for a rectangular box with length x, the girth is $2y+2z$ and the volume is xyz.

Max $f(x,y,z) = xyz$ subject to

$g(x,y,z) = x+2y+2z = 108$.

$\nabla f = (yz, xz, xy) = \lambda \nabla g = (1,2,2)$.

$yz = \lambda$ Thus $xz = 2yz = xy$

$xz = 2\lambda$ and $x = 2y = 2z$, $y = z$.

$xy = 2\lambda$

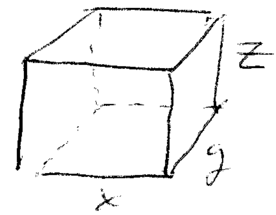
So $2z + 2z + 2z = 108$

$6z = 108$ $z = \frac{108}{6} = 18 = y$, $x = 36$

Volume $= xyz = 36 \cdot 18 \cdot 18 = 3(3/2)(3/2) = \frac{27}{4}$ ft^3

Math 2433 Exam 2 Review Solutions

86 A rectangular box without a top is to have a volume of 12 cubic feet. The materials used to construct the box cost \$3 per ft^2 for the sides and \$4 per ft^2 for the bottom. What dimensions will yield the minimum cost?



$V = xyz = 12 = g(x,y,z)$

$C = 4xy + 3\cdot(2xz + 2yz)$

$= 4xy + 6xz + 6yz$

$= f(x,y,z)$

$\nabla f = \lambda \nabla g$?

$\nabla f = (4y + 6z,\ 4x + 6z,\ 6(x+y))$

$\nabla g = (yz,\ xz,\ xy)$

So $(4y+6z,\ 4x+6z,\ 6(x+y)) = \lambda (yz,\ xz,\ xy)$

(1) $4y + 6z = \lambda yz$] mult by x — $4xy + 6xz = \lambda xyz$

(2) $4x + 6z = \lambda xz$] mult by y — $4xy + 6yz = \lambda xyz$ (2)'

(3) $6(x+y) = \lambda xy$ — So $x = y$.

mult (3) by z $\quad 6xz + 6yz = \lambda xyz$ comp (2)'

$6xz = 4xy \qquad x(6z - 4y) = 0$

$x \neq 0$ so $4y = 6z$, $x = y$. $\quad z = \frac{2}{3} y$

$xyz = y \cdot y \cdot \frac{2}{3} y = 12$

$y^3 = \frac{3}{2} \cdot 12 = 18 \qquad y = \sqrt[3]{18} = x$

$z = \frac{2}{3} y = \frac{2}{3} \sqrt[3]{2}\sqrt[3]{9} = \frac{2^{4/3}}{3^{1/3}}$.