DIFFERENTIALS AND IMPLICIT DIFFERENTIATION

DAVID H. WAGNER

1. INSTRUCTIVE EXERCISES

(1) The linear case. If $3x + 5y + 7z = 15$, solve for $z = f(x,y)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Solve for $y = g(x,z)$. Find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial z}$. Solve for $x = h(y,z)$. Find $\frac{\partial h}{\partial y}$ and $\frac{\partial h}{\partial z}$.

(2) Find the equation of the plane which is tangent to $x^3 + y^5 + z^7 = 3$ at $(1,1,1)$. For this equation, use the variables $dx = x - 1$, $dy = y - 1$, and $dz = z - 1$. Solve for $dz = f(dx, dy)$ and use this to solve for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1,1,1)$ on the surface.—they are the same as $\frac{\partial (dz)}{\partial (dx)}$ and $\frac{\partial (dz)}{\partial (dy)}$ on the tangent plane.

(3) For the same surface, point, and tangent plane as in #2, solve for $dy = g(dx, dz)$. Use this to find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$ at the point $(1,1,1)$ on the surface.

(4) For the same surface, point, and tangent plane as in #2, solve for $dx = g(dy, dz)$. Use this to find $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$ at the point $(1,1,1)$ on the surface.

(5) Failure: the linear case. If $3x + 5y = 8$, verify that $(x,y,z) = (1,1,0)$ satisfies the equation. Can you solve for $z = f(x,y)$?

(6) Failure: If $x^3 + y^5 + z^7 = 2$, verify that $(1,1,0)$ solves the equation. Find the equation of the tangent plane at this point. Use $dx = x - 1$, $dy = y - 1$, and $dz = z - 0$ as variables for this equation. Can you solve for $dz = f(dx, dy)$? Can you find $\frac{\partial z}{\partial x}$ at this point?

2. WHAT MAKES THIS WORK

**Theorem 2.1** (The Implicit Function Theorem, special case). Let $f(x,y,z)$ be continuously differentiable in a neighborhood of $(x_0, y_0, z_0)$. If $\frac{\partial f}{\partial x}(x_0, y_0, z_0) \neq 0$ then there is a neighborhood $U$ of $(y_0, z_0)$, an interval $(x_0 - \delta, x_0 + \delta)$ and a continuously differentiable function $g$ with domain $U$ and with values in $(x_0 - \delta, x_0 + \delta)$ such that all solutions of the equation $f(x,y,z) = f(x_0, y_0, z_0)$ with $x \in (x_0 - \delta, x_0 + \delta)$ and $(y,z) \in U$, lie on the graph of $g$. Moreover

$$
\frac{\partial g}{\partial y}(y_0, z_0) = \frac{\frac{\partial f}{\partial y}(x_0, y_0, z_0)}{\frac{\partial f}{\partial x}(x_0, y_0, z_0)},
$$

$$
\frac{\partial g}{\partial z}(y_0, z_0) = -\frac{\frac{\partial f}{\partial z}(x_0, y_0, z_0)}{\frac{\partial f}{\partial x}(x_0, y_0, z_0)}.
$$

**Remark 1.** The hard part of this theorem is proving the existence of the function $g$ which is “implicitly defined” by the equation. This theorem is typically proved in a course on Advanced Calculus, or in our Math 4331-32 sequence.
Remark 2. Of course, one can exchange the roles of $x$, $y$, and $z$ in this theorem. The key is that if you want, for example, $z$ to be defined implicitly as a function of $x$ and $y$, then you need $rac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$.

3. Answers

(1) (a) $z = \frac{15}{7} - \frac{3}{7}x - \frac{5}{7}y = f(x, y)$
(b) $\frac{\partial f}{\partial x} = -\frac{3}{7}$, $\frac{\partial f}{\partial y} = -\frac{5}{7}$.
(c) $y = 3 - \frac{3}{5}x - \frac{7}{5}z = g(x, z)$.
(d) $\frac{\partial g}{\partial x} = -\frac{3}{5}$, $\frac{\partial g}{\partial z} = -\frac{7}{5}$.
(e) $x = 5 - \frac{5}{3}y - \frac{7}{3}z = h(y, z)$.
(f) $\frac{\partial h}{\partial y} = -\frac{5}{3}$, $\frac{\partial h}{\partial z} = -\frac{7}{3}$.

(2) (a) $dz = -\frac{3}{7}dx - \frac{5}{7}dy = f(dx, dy)$
(b) $\frac{\partial z}{\partial x}(1, 1) = \frac{\partial f}{\partial dx}(1, 1) = -\frac{3}{7}$, $\frac{\partial z}{\partial y}(1, 1) = \frac{\partial f}{\partial dy}(1, 1) = -\frac{5}{7}$.

(3) (a) $dy = -\frac{3}{7}dx - \frac{7}{5}dz = g(dx, dz)$.
(b) $\frac{\partial y}{\partial x}(1, 1) = \frac{\partial g}{\partial dx}(1, 1) = -\frac{3}{7}$, $\frac{\partial y}{\partial z}(1, 1) = \frac{\partial g}{\partial dz}(1, 1) = -\frac{7}{5}$.

(4) (a) $dx = -\frac{5}{3}dy - \frac{7}{3}dz = h(dy, dz)$.
(b) $\frac{\partial x}{\partial y}(1, 1) = \frac{\partial h}{\partial dy}(1, 1) = -\frac{5}{3}$, $\frac{\partial x}{\partial z}(1, 1) = \frac{\partial h}{\partial dz}(1, 1) = -\frac{7}{3}$.

(5) No.

(6) (a) $3dx + 5dy + 0dz = 0$.
(b) No.