HW # 7. /20

Pb # 1. (6 pts)

$z = f(x,y)$

$f(1,4) = 3$

$\frac{\partial f}{\partial x}(1,4) = 5, \quad \frac{\partial f}{\partial y}(1,4) = 2$ 1pt

Put: $h(x,y,z) = z - f(x,y)$ 1pt

$\Rightarrow \vec{\nabla} h = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle$

The tangent plane of h @ $(1,4,3)$ is

$\vec{N} = \vec{\nabla} h(1,4,3)$ 1pt

$= \langle -5, -2, 1 \rangle$

$(T): \vec{N} \cdot \begin{pmatrix} x-1 \\ y-4 \\ z-3 \end{pmatrix} = 0$ 1pt

$\Rightarrow$

$$-5(x-1)-2(y-4)+z-3=0$$

$$\Rightarrow \quad x = -\frac{2}{5}(y-4)+\frac{z}{5}-\frac{3}{5}+1$$

$$= -\frac{2}{5}(y-4)+\frac{z}{5}+\frac{2}{5}$$

1pt

$$\Rightarrow \boxed{\frac{\partial x}{\partial y} = \frac{\partial g}{\partial y} = -\frac{2}{5}}$$

1pt

HW#7. /20

Pb#2 (8 pts)

$f(x,y) = xy$

$g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} = 1$

Applying Lagrange's method

$\nabla f = \lambda \nabla g$ ... 1pt

$\Leftrightarrow \nabla f \times \nabla g = \langle 0, 0, 0 \rangle$

$\vec{\nabla} f = \langle y, x \rangle$

$\vec{\nabla} g = \langle \frac{x}{2}, \frac{2}{9} y \rangle$

$$\vec{\nabla} f \times \vec{\nabla} g = \begin{vmatrix} i & j & k \\ y & x & 0 \\ \frac{x}{2} & \frac{2}{9}y & 0 \end{vmatrix}$$

$= \left(\frac{2}{9} y^2 - \frac{x^2}{2}\right) k$ 1pt

$= \langle 0, 0, 0 \rangle$

$$\Rightarrow \begin{cases} \frac{2}{9}y^2 - \frac{x^2}{2} = 0 \quad \cdots (1) \\ \frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \cdots (2) \end{cases}$$

From (1): $\boxed{\frac{x^2}{4} = \frac{y^2}{9}}$ — 1pt

Plug into (2):

$2\frac{y^2}{9} = 1 \Rightarrow \boxed{y = \pm\frac{3}{\sqrt{2}}}$ — 1pt

For $y = -3/\sqrt{2}$

$$x^2 = \frac{4}{9}\left(-3/\sqrt{2}\right)^2$$
$$= 2$$

$\Rightarrow \boxed{x = \pm\sqrt{2}}$ — 1pt

$\Rightarrow \left(-\sqrt{2}, -3/\sqrt{2}\right)$;

$\left(\sqrt{2}, -3/\sqrt{2}\right)$

HW#7. /20

Pb #2 (8 pts)

for $y = \frac{3}{\sqrt{2}}$

$\Rightarrow x^2 = \frac{4}{9}\left(\frac{3}{\sqrt{2}}\right)^2 = 2$

$\Rightarrow \boxed{x = \pm\sqrt{2}}$

$\Rightarrow \left(-\sqrt{2}, \frac{3}{\sqrt{2}}\right)$
$\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ 1 pt

$f(x,y) = xy$

$f\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right) = f\left(-\sqrt{2}, -\frac{3}{\sqrt{2}}\right)$
$= 3 > 0$

$f\left(-\sqrt{2}, \frac{3}{\sqrt{2}}\right) = f\left(\sqrt{2}, \frac{-3}{\sqrt{2}}\right)$
$= -3 < 0$

$\Rightarrow f$ has a max
@ $\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ and
$\left(-\sqrt{2}, -3/\sqrt{2}\right)$ 1pt

And f has a min

@ $\left(-\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ - - - 1pt

and $\left(\sqrt{2}, -\frac{3}{\sqrt{2}}\right)$

Pb #3 : 6 pts

(a)

$V(x,y) = \underbrace{2xe^{3y}}_{P(x,y)}\, i + \underbrace{(3x^2e^{3y}+5)}_{Q(x,y)}\, j$

By theorem,

$V = \nabla f$ if and only if

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$? ... 1 pt

$\frac{\partial P}{\partial y} = 6xe^{3y}$ ... 1 pt

$\frac{\partial Q}{\partial x} = 6xe^{3y} = \frac{\partial P}{\partial y}$

$\Rightarrow V = \nabla f$ ... (1 pt)

And

$\begin{cases} \frac{\partial f}{\partial x} = P \quad \text{... (1)} \\ \frac{\partial f}{\partial y} = Q \quad \text{... (2)} \end{cases}$

From (1)

$$f(x,y) = \int P(x,y)\,dx$$

$$= \int 2x\,e^{3y}\,dx$$

$$= e^{3y}x^2 + h(y) \quad \cdots (*)$$ 1 pt

Verification

$$\frac{\partial f}{\partial x} = 2x\,e^{3y} = P \quad \checkmark$$

Using (*) and (2)

$$\frac{\partial f}{\partial y} = \cancel{3x^2 e^{3y}} + h'(y)$$

$$= \cancel{3x^2 e^{3y}} + 5 \cdots$$ 1 pt

$$\Rightarrow h'(y) = 5$$

$$\Rightarrow \boxed{h(y) = 5y + C \quad C \in \mathbb{R}}$$ 1 pt

$$\Rightarrow \boxed{f(x,y) = x^2 e^{3y} + 5y + C \quad C \in \mathbb{R}}$$

b #3 : 6 pts

b

$$u(x,y) = \overbrace{2y e^{3x}}^{P}\, i + \overbrace{(3y^2 e^{3x} + 5)}^{Q}\, j$$

$\vec{u}$ is the gradient of a function f

i.e $\vec{u} = \nabla f$

iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = 2e^{3x}$$

$$\frac{\partial Q}{\partial x} = 9y^2 e^{3x} \neq \frac{\partial P}{\partial y}$$

$\Rightarrow \vec{u}$ is not the gradient of a fct.

1 pt