Show all work!

1. Evaluate the integral of $f(x,y) = e^{-(x+y)}$ over the domain $\Omega = \{(x,y) : 0 \le x,\ 0 \le y,\ x+y \le z\}$.
Solution: Ω can also be described with the inequalities $0 \le x \le z$, $0 \le y \le z-x$. Then

$$\begin{aligned}\iint_\Omega e^{-(x+y)} dxdy &= \int_0^z \int_0^{z-x} e^{-(x+y)} dy\ dx \\ &= \int_{x=0}^{z} e^{-x}\left(-e^{-y}\right)\Big|_{y=0}^{z-x}\ dx \\ &= \int_{x=0}^{z} e^{-x}\left(1 - e^{x-z}\right)\ dx \\ &= \int_{x=0}^{z} e^{-x} - e^{-z}\ dx \\ &= -e^{-x} - xe^{-z}\Big|_{x=0}^{z} = 1 - e^{-z} - ze^{-z}.\end{aligned}$$

Remark 1 *The function* $f(x,y) = \begin{cases} e^{-(x+y)} & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise}\end{cases}$ *is a joint probability density function for two independent exponential random variables* X, Y *with mean 1. The calculation above computes the probability that* $X+Y \le z$. *The result is the cumulative distribution function for* $Z = X+Y$. *The derivative of the result is the probability density function function for* Z.

2. Reverse the order of integration and evaluate:

$$\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \sin\left(x^2\right)\ dx\ dy \tag{1}$$

Solution The domain of integration is $\left\{(x,y) : 0 \le y \le \sqrt{\pi/2},\ y \le x \le \sqrt{\pi/2}\right\}$. This domain can also be described as $\left\{(x,y) : 0 \le x \le \sqrt{\pi/2},\ 0 \le y \le x\right\}$. Thus, reversing the order of integration gives

$$\begin{aligned}\int_{x=0}^{\sqrt{\pi/2}} \int_{y=0}^{x} \sin\left(x^2\right)\ dy\ dx &= \int_{x=0}^{\sqrt{\pi/2}} y \sin\left(x^2\right)\Big|_{y=0}^{x} dx \\ &= \int_{x=0}^{\sqrt{\pi/2}} x\sin\left(x^2\right)\ dx = \int_{u=0}^{\pi/2} \sin(u)\ \frac{du}{2} = \frac{1}{2}.\end{aligned}$$

3. Integrate $f(x,y) = e^{-\left(x^2+y^2\right)}$ over the annular region $\ln(2) \leq x^2 + y^2 \leq \ln(5)$.
Solution In polar coordinates, the integral is

$$\int_{\theta=0}^{2\pi} \int_{r=\sqrt{\ln(2)}}^{\sqrt{\ln 5}} e^{-r^2}\ r\ dr\ d\theta = 2\pi \int_{u=\ln(2)}^{\ln(5)} e^{-u}\ \frac{du}{2} = \pi\left(e^{-\ln(2)} - e^{-\ln(5)}\right)$$
$$= \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}.$$