Pb # 1. (2 pts) TA: Youssef Menacer $\frac{\ \ }{40}$

Jacobian.

$x = aAu + Bv$
$y = Cu + Dv$

$dx\,dy = J\,du\,dv = (aAD - BC)\,du\,dv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} aA & B \\ C & D \end{vmatrix} = \boxed{aAD - BC}$$

Pb # 2. (2 pts)

For θ fixed,

$x = u\cos(\theta) - v\sin(\theta)$, $dx\,dy = J\,du\,dv = \boxed{du\,dv}$
$y = u\sin(\theta) + v\cos(\theta)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} = \cos^2 + \sin^2 = \boxed{1}$$

Pb # 3 (2 pts)

$x = uv$
$y = u^2 + v^2$, $dx\,dy = J\,du\,dv = 2(u^2 + v^2)\,du\,dv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 2u & 2v \end{vmatrix} = 2(v^2 - u^2)$$

Pb # 4. (4 pts)

$\Omega = \{(x,y) \mid 0 \le x - 3y \le 2 ;\ 0 \le x + 3y \le 2\}$. $I = \iint_\Omega (x^2 - 9y^2)\,dx\,dy$

Using pb # 1,

$u = x - 3y$
$v = x + 3y$ $\Big\downarrow \oplus \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$ $\Rightarrow J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{6} \end{vmatrix} = \frac{1}{12} + \frac{1}{12} = \boxed{\frac{1}{6}}$

$v = x + 3y$
$u = x - 3y$ $\Big\downarrow \ominus \Rightarrow y = \frac{1}{6}v - \frac{1}{6}u$

$\boxed{y = -\frac{1}{6}u + \frac{1}{6}v}$

$0 \le u \le 2$
$0 \le v \le 2$

$$\Rightarrow I = \iint_\Omega (x^2 - 9y^2)\,dx\,dy = \frac{1}{6}\int_0^2\int_0^2 \left(\frac{1}{4}(u+v)^2 - \frac{9}{36}(u-v)^2\right)du\,dv$$

with $\frac{9}{36} = \frac{1}{4}$.

$$= \frac{1}{6}\cdot\frac{1}{4}\int_0^2\int_0^2 \Big(\underbrace{(u+v)^2 - (u-v)^2}_{4uv}\Big)du\,dv = \frac{1}{6}\left(\int_0^2 u\,du\right)\left(\int_0^2 v\,dv\right) \quad \frac{1}{6}\cdot\frac{1}{4}\,2^4 = \boxed{\frac{4}{6}}$$

$= \boxed{\frac{2}{3}}$

Pb # 5 (3 pts)

$h(x,y) = xy\,i + x\,j \quad , \quad r(t) = \overset{x}{\overline{t}}\,i + \overset{y}{\overbrace{(2-2t)}}\,j$
$0 \le t \le 1$

$$\int_C h\cdot dr = \int_0^1 h(r(t))\cdot r'(t)\,dt = \int_0^1 \left(t(2-2t)\,i + t\,j\right)\cdot\left(i - 2j\right)dt$$

$$= \int_0^1 \left(2(t-t^2) - 2t\right)dt = \boxed{-\frac{2}{3}}$$

Pb # 6. (3 pts)

$h(x,y) = xy\,i + x\,j \;/\; r(t) = \overset{x}{\overline{t}}\,i + \overset{y}{\overbrace{2(1-t^2)}}\,j, \quad 0 \le t \le 1$

$$\int_C h\cdot dr = \int_0^1 h(r(t))\cdot r'(t)\,dt = \int\left(2t(1-t^2)\,i + t\,j\right)\cdot\left(i - 4t\,j\right)dt$$

$$= \int_0^1 \left(2t - 2t^3 - 4t^2\right)dt = t^2 - \frac{1}{2}t^4 - \frac{4}{3}t^3\Big|_{t=0}^{1} = 1 - \frac{1}{2} - \frac{4}{3}$$

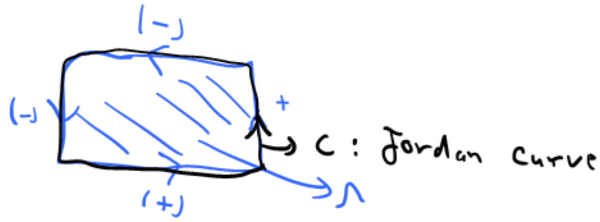
$$= \frac{1}{2} - \frac{4}{3} = \boxed{\frac{-5}{6}}$$

Pb # 7. (4 pts)

Clockwise $(-)$, Counterclockwise $(+)$

$$\int_C h(x,y)\cdot dr = \int_C \left((2xy^2 - x)\,i + (2x^2y + 3x)\,j\right)\cdot dr$$

$\Omega : \; 0 \le x \le 2, \quad 0 \le y \le 3$



$r(t) = x(t)\,i + y(t)\,j$

$f(x,y) = P(x,y)\,i + Q(x,y)\,j$, such that

$P(x,y) = 2xy^2 - x \Rightarrow \frac{\partial P}{\partial y} = 4xy$

$Q(x,y) = 2x^2y + 3x \Rightarrow \frac{\partial Q}{\partial x} = 4xy + 3$

$$\xRightarrow[\text{Green's thm}]{\text{Apply}} \int_C f\cdot dr = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$= 3\underbrace{\iint_\Omega dx\,dy}_{\text{= Area of the rectangle } 2\times 3} = 3(2\times 3) = \boxed{18}$$

Pb # 8. (4 pts)

$C: \; r(t) = \overset{x}{\overline{2t}}\,i + \overset{y}{\overline{(1-t)^2}}\,j \;, \quad 0 \le t \le 1 \Rightarrow r'(t) = 2i - 2t\,j \;/\; \begin{matrix} r(0) = (0,1) \\ r(1) = (2,0)\end{matrix}$

$$W = \text{work done by } F = \int_C F(x,y)\cdot dr = \int_C \nabla f\cdot dr = \boxed{f(r(1)) - f(r(0)) = 0}$$

$F(x,y) = \underbrace{3x^2y}_{P}\,i + \underbrace{x^3}_{Q}\,j \Rightarrow \frac{\partial Q}{\partial x} = 3x^2 = \frac{\partial P}{\partial y} \Rightarrow \boxed{F = \nabla f}$

$f_x = P = 3x^2y \Rightarrow \boxed{f(x,y) = x^3y + c}$ $c \in \mathbb{R}$: $f(2,0) - f(0,1) = c - c = \boxed{0}$

Pb 9. (3 pts)

$h(x,y) = P\,i + Q\,j$, where

$P(x,y) = \sin(x) + 2y \Rightarrow \frac{\partial P}{\partial y} = 2$

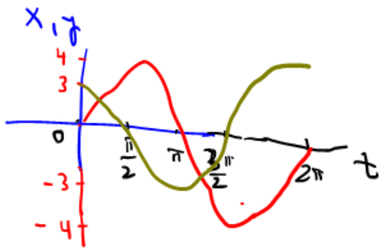
$Q(x,y) = \cos(y) + 2x \Rightarrow \frac{\partial Q}{\partial x} = 2 = \frac{\partial P}{\partial y} \Rightarrow \boxed{h = \nabla f}$

$r(t) = \underbrace{3\cos(t)}_{x}\, i + \underbrace{4\sin(t)}_{y}\, j$

$0 \le t \le 2\pi$

C : Jordan curve

ellipse



$$\int_C h \cdot dr = \int_C P\,dx + Q\,dy = \int_C \nabla f \cdot dr = 0$$

Pb 10. (3 pts)

$h(x,y) = (\underbrace{\sin(x) + 2y}_{P})\,i + (\underbrace{\cos(y) - 3x}_{Q})\,j$, $r(t) = \underbrace{3\cos(t)}_{x(t)}\, i + \underbrace{4\sin(t)}_{y(t)}\, j$

$0 \le t \le 2\pi$

$\frac{\partial P}{\partial y} = 2 \neq \frac{\partial Q}{\partial x} = -3$

C: ellipse

Ω: Jordan

$$\int_C h \cdot dr = \int_0^{2\pi} h(r)\cdot r'(t)\,dt = \int_0^{2\pi} (P\,i + Q\,j)(x'(t) + y'(t))\,dt = \int_0^{2\pi} P\,dx + Q\,dy$$

$x(t) = 3\cos(t)$

$y(t) = 4\sin(t)$

$$\underset{\text{Green's thm}}{=} \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_\Omega (-3 - 2)\,dx\,dy = -5 \underbrace{\iint_\Omega dx\,dy}_{\text{Area of an ellipse} = a\cdot b\cdot\pi} = -5(3\cdot 4)\pi = \boxed{-60\pi}$$

$a = 3$, $b = 4$

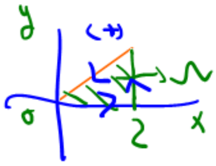
Pb # 11 (4 pts)

(+) ↺

$\oint_C y^2 dx + xy\,dy$

$\Omega: 0 \le x \le 2$

$0 \le y \le \frac{3x}{2}$



Ω is a Jordan region $\Rightarrow$ We apply Green's thm

$$\oint_C \underbrace{y^2}_{P} dx + \underbrace{xy}_{Q}\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^2 \int_0^{\frac{3x}{2}} (y - 2y)\,dy\,dx = -\int_0^2 \frac{y^2}{2}\Big|_{y=0}^{\frac{3x}{2}} dx$$

$$= -\frac{1}{2}\left(\frac{3}{2}\right)^2 \frac{1}{3}\, \cancel{2}^3 = \boxed{-3}$$

Pb #12 (6 pts) (3 pts (a), 3 pts (b))

C is a piece wise-smooth Jordan

Counterwise direction

(a)

(0,0) included

(a)

$$I = \oint_C \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx = \oint_C \overbrace{\frac{-y}{x^2+y^2}}^{P}\,dx + \overbrace{\frac{x}{x^2+y^2}}^{Q}\,dy$$

Apply Green's thm

$$\Rightarrow \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dx\,dy = \boxed{0}$$

$$P = \frac{-y}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$Q = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial Q}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial P}{\partial y}$$

(b) Since C includes (0,0)

$\Rightarrow C = C_1 \cup C_2$

where ε is a very small number (e.g. $\varepsilon = 10^{-6}$)

$$\Rightarrow \int_C P\,dx + Q\,dy = \int_{C_1 \cup C_2} P\,dx + Q\,dy = \underbrace{\int_{C_1} P\,dx + Q\,dy}_{=0 \text{ from (a)}} + \int_{C_2} P\,dx + Q\,dy$$

$$= \int_{C_2} P\,dx + Q\,dy$$

Now, for circle $C((0,0),\varepsilon)$, put $\begin{cases} x = \varepsilon\cos(t) \\ y = \varepsilon\sin(t) \end{cases}$ $0 \le t \le 2\pi$

$$\Rightarrow \int_C P\,dx + Q\,dy = \int_{C_2} P\,dx + dy = \int_0^{2\pi} \frac{1}{\varepsilon^2}\left(\varepsilon^2\cos^2(t) + \varepsilon^2\sin^2(t)\right)dt = \int_0^{2\pi} dt = \boxed{2\pi}$$