Math 3334
October 28, 2019 Exam 2 Review

1. Suppose that f is a function on $\mathbb{R}$ such that $|f(b)-f(a)| \leq M|b-a|^2$ for all $a,\ b \in \mathbb{R}$. Prove that f is a constant function. (20)
Solution: $|f'(x)| = \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \leq \lim_{h\to 0}\left|\frac{h^2}{h}\right| = 0$ for every $x \in \mathbb{R}$. So f is constant.

2. Define *Lipschitz continous.* (8)
Solution: A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous if there is $K \in \mathbb{R}$ such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x}-\mathbf{y}\|$$

for all $\mathbf{x},\ \mathbf{y} \in D$.

3. Let U be a convex open set in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be differentiable on U with $\|Df(x,y)\| \leq 5$ for all $(x,y) \in U$. Prove that f is Lipschitz continuous on U. (17)

Proof. Let $p,\ q$ be points in U. Since U is convex, the line segment from p to q is contained in U. Then by the Mean Value Theorem there is a point c on this line segment such that $f(q)-f(p) = Df(c)(q-p) = \nabla f(c)\cdot(q-p)$. By the Schwartz inequality,

$$|f(q)-f(p)| \leq \|\nabla f(c)\|\,\|q-p\| \leq 5\,\|q-p\|\,.$$

Thus, f is Lipschitz continuous on U. □

4. Let $f(x,y) = \int_0^{x+2y} e^{-t^2}\ dt$.
(a) Find $Df(x,y)$ (10)
Solution: $Df(x,y)(h,k) = e^{-(x+2y)^2}(h+2k)$.

(b) Find the directional derivative of f at $(3,1)$, $D_\beta f(3,1)$, where $\beta = (3,4)/5$. (10)
Solution: $D_\beta f(3,1) = Df(3,1)(\beta) = e^{-(3+2)^2}\left(\frac{3}{5} + 2\cdot\frac{4}{5}\right) = e^{-25}\left(\frac{11}{5}\right)$.

(c) Find a unit vector β which maximizes $D_\beta f(3,1)$, and find the maximum value. (10)
Solution:

$$\beta = \frac{\nabla f(3,1)}{\|\nabla f(3,1)\|} = \frac{e^{-25}(1,2)}{\|e^{-25}(1,2)\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

Then, $D_\beta f(3,1) = \|\nabla f(3,1)\| = e^{-25}\sqrt{5}$.

5. Suppose f is a C^1 function on an open set U in $\mathbb{R}^2$, and $z = f(x,y)$, $Df(x,y) = (p(x,y),\ T(x,y))$. If $x = g(y,z)$, find $Dg(y,z)$. What simple condition on p and/or T is necessary for this computation? (25)
Solution:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = p(x,y)dx + T(x,y)dy.$$

Then

$$dx = \frac{1}{p(x,y)}dz - \frac{T(x,y)}{p(x,y)}.$$

So

$$\nabla G(y,z(x,y)) = \left(\frac{1}{p(x,y)},\ \frac{T(x,y)}{p(x,y)}\right).$$

Or,

$$\nabla G(y,z) = \left(\frac{1}{p(G(y,z),y)},\ \frac{T(G(y,z),y)}{p(G(y,z),y)}\right)$$

To do this we need $p(x,y) \neq 0$.