Math 3334
December 10, 2018 Final Exam Name ____________________

1. Find a Taylor polynomial with just enough terms to approximate $f(x) = \cos(2x)$ within 0.01 on $-\frac{1}{2} \le x \le \frac{1}{2}$. (18)

2. Use the definition of limit to show that (20)

$$\lim_{(x,y)\to(1,2)} 2x^2 - 3y^2 = -10.$$

3. A smooth function $f(x, y)$ has $\nabla f(2, 3) = 0$ and second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}(2,3) = A, \quad \frac{\partial^2 f}{\partial x \partial y}(2,3) = 4, \quad \frac{\partial^2 f}{\partial y^2}(2,3) = -2.$$

For what values of A is $f(2, 3)$

(a) A local minimum? (10)

(b) Neither a local minimum nor a local maximum? (10)

(c) A local maximum? (10)

4. Determine whether the equation

$$xy - y\ln(z) + \sin(xz) = 0$$

(a) has a solution $z = f(x, y)$ for (x, y, z) near $(0, 2, 1)$. If so, compute $\nabla f(0, 2)$. (12)

(b) has a solution $x = g(y, z)$ for (x, y, z) near $(0, 2, 1)$. If so, compute $\nabla g(2, 1)$. (12)

(c) has a solution $y = h(x, z)$ for (x, y, z) near $(0, 2, 1)$. If so, compute $\nabla h(0, 1)$. (12)

5. (a) Let (15)

$$f(x) = \begin{cases} x\sin(1/x). & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Prove from the definition that $f(x)$ is not differentiable at $x = 0$.

(b) Let (15)

$$g(x) = \begin{cases} x^2\sin(1/x). & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Prove from the definition that $g(x)$ is differentiable at $x = 0$, and find $g'(0)$.

6. Let $P = \{(x, y) : \ 0 \le 2y - x \le 4, \ 0 \le 3x - y \le 5\}$. Evaluate (20)

$$\iint_P 2y - x\,dx\,dy$$

Hint: Use a change of variables that transforms P into a rectangle $a \le u \le b$, $c \le v \le d$.

7. Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}$. (16)

8. Let $P(x, y)$, $Q(x, y)$ be continuously differentiable on an open rectangle $A \subset \mathbb{R}^2$. Suppose (20)

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$$

for $(x, y) \in A$. Suppose $(a, b) \in A$ and let

$$f(x,y) = \int_a^x P(t,b)\,dt + \int_b^y Q(x,t)\,dt.$$

Prove that $\frac{\partial f}{\partial x}(x,y) = P(x,y)$ and that $\frac{\partial f}{\partial y}(x,y) = Q(x,y)$.