Math 3363 Final Exam Name___________________

Review sheet 1

1. Which of the following functions, when extended as 2π periodic functions, are equal to their Fourier series (for all x)? (Hint: **DO NOT** compute Fourier coefficients).

a.	$f(x) = \sin(x/2)$, $-\pi < x \le \pi$		6 pts
b.	$f(x) = \sin(x)$, $-\pi < x \le \pi$		6 pts
c.	$f(x) = \cos(x)$, $-\pi < x \le \pi$		6 pts
d.	$f(x) = \cos(x/2)$, $-\pi < x \le \pi$		6 pts
e.	$f(x) = \pi - x$	$-\pi < x \le \pi$	4 pts
f.	$f(x) = 1 - \lvert x \rvert$,	$-\pi < x \le \pi$	4 pts
g.	$f(x) = \begin{cases} \pi - x, & 0 < x \le \pi \\ -x - \pi, & -\pi < x \le 0 \end{cases}$,	$-\pi < x \le \pi$	8 pts
h.	$f(x) = \begin{cases} x - \pi/2, & \pi/2 < x \le \pi \\ \frac{1}{3}(\pi/2 - x), & -\pi < x \le \pi/2 \end{cases}$,	$-\pi < x \le \pi$	8 pts

2. If $f(x) = e^{-|x|}$, find the Fourier transform of f. Show your work! 20 pts

3. a. Let $f(x) = e^{-4x^2}$. Note that $f'(x) = -8xf(x)$. Use this and the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, to derive the Fourier Transform of f. 20 pts

b. If $f(x) = e^{-3(x+2)^2}$, find the Fourier transform of f using the attached table. 20 pts

4. A steel rod is removed from an oven at t = 0 with a temperature of 1200. The left end (x=0) is put in an ice bath, the length of the rod and the right end (x=π) are insulated. The temperature u(x,t) satisfies:

$$\frac{\partial u}{\partial t} = 7\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = 1200$$

Estimate the maximum temperature of the rod when t = 8. 20 pts.

5. Suppose Ω is a bounded domain in the xy plane, and the solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0, \qquad u(x,y) = 0 \text{ if } (x,y) \text{ is in the boundary of } \Omega,$$

are $(\lambda, u(x,y)) = (\lambda_n, \phi_n(x,y))$, with $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$.
You have determined that $\lambda_1 = 5$ and $\lambda_2 = 8$.

a. Find the special solutions of:

$$\frac{\partial^2 u}{\partial t^2} = 9\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \ (x,y) \in \Omega, \ 0 < t$$

14 pts

with homogeneous Dirichlet boundary conditions, which correspond to λ_1 and λ_2. If t is measured in seconds, what is the frequency of vibration for these special solutions?

6. A round flat plate of radius 3 is insulated on its surfaces and has an equilibrium heat distribution $u(r,\theta)$. You are able to measure the temperature only on the boundary. You have found that $u(3,\theta) = 2 - |\theta|,\ -\pi \le \theta < \pi$.

a. What is the temperature at the center? 12 pts

b. What is the maximum temperature on the plate? 12 pts

7. Find the solution to the Laplace equation in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ r < 5,\ 0 < \theta\ ''\ 2\pi.$$

with boundary condition:

$u(5,\theta) = \cos(3\theta)$, $0 < \theta \le 2\pi$. 20 pts

(Reminder: to solve $ar^2\phi''(r) + br\phi'(r) + c\phi(r) = 0$, try $\phi = r^{\alpha}$).

8. Which of the following eigenvalue problems have only positive eigenvalues λ?

a. $\varphi'' + 3\lambda\phi + \phi = 0,\ \ \phi(0) = \phi(2) = 0$ 10 pts

b. $\varphi'' + 3\lambda\phi + \phi = 0,\ \ \phi(0) = \phi(4) = 0$ 10 pts

9. Consider the differential equation:

$$r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + \left(\lambda r^2 - m^2\right)f = 0$$

If we couple this equation with boundary conditions:

f(5) = 0, and f is bounded as $z \to 0$,

the combined problem has a sequence of eigenfunctions $f_n(r)$ with eigenvalues λ_n,

n = 1, 2, 3, …

a. How many zeros does the nth eigenfunction have, strictly between 0 and 5? 12 pts

b. What equation do the eigenvalues satisfy? 12 pts

c. What is the orthogonality relation is satisfied by the eigenfunctions? 14 pts

10. A certain non-isotropic material has a heat conductivity in the y direction that is 4 times the heat conductivity in the x direction, so that the equation for heat conduction is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} \quad -\infty < x < \infty,\ 0 \le y < \infty$$ 27 pts

At equilibrium we measure the temperature distribution along the x-axis to be:

$$u(x,0) = e^{-5x^2},\quad -\infty < x < \infty.$$

Find the equilibrium heat distribution in the half plane . $-\infty < x < \infty,\ 0 \le y < \infty$ $\left(\frac{\partial u}{\partial t} = 0\right)$. (*Hint: Take the Fourier transform in x.)*

11. A uranium rod of length π is initially at $0°C$. The ends of the rod sit in ice baths at $0°C$. Nuclear fission adds heat to the rod in a uniform manner, so that the temperature of the rod, u(x,t), satisfies:

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} + 1,\quad u(0,t) = u(\pi,t) = 0,\quad u(x,0) = 0$$

a. Find an equilibrium temperature distribution v(x) for this problem. 10 pts.

b. Use v(x) to transform the problem into one with a homogeneous PDE. 12 pts

New unknown w, in terms of u and v:

PDE for w:

Initial Condition:

Boundary Conditions:

c. Solve the transformed problem in (b). Do not solve for the Fourier coefficients, just show the formula. Use the result to solve the original problem.

12. Consider a rectangular metal plate, with coordinates 0< x < 2, 0 < y < 3. The temperature distribution u(x,y,t) of the plate obeys:

$$\frac{\partial u}{\partial t} = 5\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\quad \frac{\partial u}{\partial x}(x,y,t) = 0 \text{ if x = 0 or 2, u(x,y,t) = 0 if y = 0 or 3.}$$

a. Find the smallest eigenvalue λ_1 of the problem: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0,\ (x,y) \in \Omega$,

with boundary conditions $\frac{\partial u}{\partial x}(x,y) = 0$ if x = 0 or 2, u(x,y) = 0 if y = 0 or 3, where Ω is the rectangle 0< x < 2, 0 < y < 3. 15 pts

b. Find a solution to the heat equation above, corresponding to λ_1. What is the ratio $\left(\max_{(x,y)\in\Omega} u(x,y,0)\right) \Big/ \left(\max_{(x,y)\in\Omega} u(x,y,\ln 2)\right)$ for this solution? 15 pts

13. Solve the circularly symmetric heat equation:

$$\frac{\partial u}{\partial t} = \frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),\ 0 < r < 3,\ 0 < t,\ u(r = 3, t) = 0, \text{ with the initial condition}$$

$$u(r,0) = 3 - r,\ 0 < r \le 3.$$

a. $$u(r,t) = \sum_{n=1}^{\infty} A_n T_n(t)\phi_n(r)$$

Identify the eigenfunctions $\phi_n(r)$, the eigenvalues λ_n, and the functions $T_n(t)$.

10 pts

b. Give an equation for A_n in terms of ϕ_n. 10 pts

c. What is the orthogonality relation between ϕ_n and ϕ_j? 15 pts

14. a. Find a Green's function G(x,s) for the problem: $u''(x) = f(x),\ 0 < x < 1,\ u(0) = u'(1) = 0.$

15 pts

b. Express the solution to: $u''(x) = f(x),\ 0 < x < 1,\ u(0) = u'(1) = 0$
in terms of the Green's function found in part a. Verify that the solution solves the differential equation and boundary conditions. 15 pts

15. Solve the diffusion equation with convection:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}, \quad -\infty < x < \infty, \quad u(x,0) = f(x)$$

Hint: Take the Fourier transform with respect to x. Use the convolution theorem and the shift theorem. 30 pts

16. A fuel line on a fuel-injected car engine is subject to vibrations from the engine. The displacement u(x,t) from its undisturbed configuration satisfies:

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = \sin(\omega t)\sin(x),\ 0 < x < \pi,\ u(0,t) = u(\pi,t) = 0.$$

Assume $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0.$

a. If $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\phi_n(x)$, what are the eigenfunctions $\phi_n(x)$ and the eigenvalues λ_n?

10 pts

b. What differential equations and initial conditions are satisfied by $B_n(t)$? 12 pts

c. Solve for $B_n(t)$. For what positive value of ω does the solution blow up? 14 pts