1. Which of the following functions, when extended as 2π periodic functions, are equal to their Fourier series (for all x)? (Hint: **DO NOT** compute Fourier coefficients).

a. $f(x) = \sin(x/2)$, $-\pi < x \le \pi$ **No** 6 pts

b. $f(x) = \sin(x)$, $-\pi < x \le \pi$ **Yes** 6 pts

c. $f(x) = \cos(x)$, $-\pi < x \le \pi$ **Yes** 6 pts

d. $f(x) = \cos(x/2)$, $-\pi < x \le \pi$ **Yes** 6 pts

e. $f(x) = \pi - x$ $-\pi < x \le \pi$ **No** 4 pts

f. $f(x) = 1 - |x|$, $-\pi < x \le \pi$ **Yes** 4 pts

g. $f(x) = \begin{cases} \pi - x, & 0 < x \le \pi \\ -x - \pi, & -\pi < x \le 0 \end{cases}$, **No** 8 pts

h. $f(x) = \begin{cases} x - \pi/2, & \pi/2 < x \le \pi \\ \frac{1}{3}(\pi/2 - x), & -\pi < x \le \pi/2 \end{cases}$, **Yes** 8 pts

2. If $f(x) = e^{-|x|}$, find the Fourier transform of f. Show your work! **See textbook** 20 pts

3. a. Let $f(x) = e^{-4x^2}$. Note that $f'(x) = -8xf(x)$. Use this and the fact that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, to derive the Fourier Transform of f. **See textbook** 20 pts

b. If $f(x) = e^{-3(x+2)^2}$, find the Fourier transform of f using the attached table. 20 pts

$$\hat{f}(\omega) = \frac{1}{\sqrt{12\pi}} e^{-i2\omega} e^{-\omega^2/12}$$

4. A steel rod is removed from an oven at t = 0 with a temperature of 1200. The left end (x=0) is put in an ice bath, the length of the rod and the right end (x=π) are insulated. The temperature u(x,t) satisfies:

$$\frac{\partial u}{\partial t} = 7\frac{\partial^2 u}{\partial x^2},\quad u(0,t) = 0,\quad \frac{\partial u}{\partial x}(\pi,t) = 0,\quad u(x,0) = 1200$$

Estimate the maximum temperature of the rod when t = 8. 20 pts.

$$\max_{0<x<\pi} u(x,8) = u(\pi,8) \cong \text{value of first term at } (\pi,8)$$

$$= \frac{4800}{\pi} e^{-14}$$

5. Suppose Ω is a bounded domain in the xy plane, and the solutions of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\lambda u=0, \qquad u(x,y)=0 \text{ if } (x,y) \text{ is in the boundary of } \Omega,$$

are $(\lambda,\, u(x,y))=(\lambda_n,\, \phi_n(x,y))$, with $0<\lambda_1<\lambda_2<\ldots<\lambda_n<\ldots$.
You have determined that $\lambda_1=5$ and $\lambda_2=8$.

a. Find the special solutions of:

$$\frac{\partial^2 u}{\partial t^2}=9\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\ (x,y)\in\Omega,\ 0<t$$ 14 pts

with homogeneous Dirichlet boundary conditions, which correspond to λ_1 and λ_2. If t is measured in seconds, what is the frequency of vibration for these special solutions?

$$u_1(x,y,t)=\phi_1(x,y)\left(A_1\cos\left(3\sqrt{5}t\right)+B_1\sin\left(3\sqrt{5}t\right)\right),\ u_2(x,y,t)=\phi_2(x,y)\left(A_2\cos\left(3\sqrt{8}t\right)+B_2\sin\left(3\sqrt{8}t\right)\right)$$

Frequencies $\omega_1=\dfrac{3\sqrt{5}}{2\pi},\ \omega_2=\dfrac{3\sqrt{8}}{2\pi}$

6. A round flat plate of radius 3 is insulated on its surfaces and has an equilibrium heat distribution $u(r,\theta)$. You are able to measure the temperature only on the boundary. You have found that $u(3,\theta)=2-|\theta|,\ -\pi\le\theta<\pi$.

a. What is the temperature at the center? 12 pts

= average of temperature on rim = $2-\dfrac{\pi}{2}$

b. What is the maximum temperature on the plate? 12 pts

= maximum temperature on rim = 2.

7. Find the solution to the Laplace equation in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0,\ r<5,\ 0<\theta \text{ ″ } 2\pi.$$

with boundary condition:

$u(5,\theta)=\cos(3\theta)$, $0<\theta\le 2\pi$. 20 pts

(Reminder: to solve $ar^2\phi''(r)+br\phi'(r)+c\phi(r)=0$, try $\phi=r^\alpha$).

$$u(r,\theta)=\left(\frac{r}{5}\right)^3\cos\left(3\theta\right)$$

8. Which of the following eigenvalue problems have only positive eigenvalues λ?

a. $\varphi'' + 3\lambda\phi + \phi = 0, \quad \phi(0) = \phi(2) = 0$ 10 pts

$\lambda_1 = \frac{1}{3}\left(\left(\frac{\pi}{2}\right)^2 - 1\right) > 0.$ All eigenvalues λ are positive.

b. $\varphi'' + 3\lambda\phi + \phi = 0, \quad \phi(0) = \phi(4) = 0$ 10 pts

$\lambda_1 = \frac{1}{3}\left(\left(\frac{\pi}{4}\right)^2 - 1\right) < 0.$ One eigenvalue λ is negative.

9. Consider the differential equation:

$$r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + \left(\lambda r^2 - m^2\right)f = 0$$

If we couple this equation with boundary conditions:

f(5) = 0, and f is bounded as $z \to 0$,

the combined problem has a sequence of eigenfunctions $f_n(r)$ with eigenvalues λ_n,

n = 1, 2, 3, …

a. How many zeros does the nth eigenfunction have, strictly between 0 and 5? 12 pts

n-1

b. What equation do the eigenvalues satisfy? 12 pts

$J_m\left(\sqrt{\lambda_{mn}}\,5\right) = 0,$ or $\lambda_{mn} = \left(\frac{z_{mn}}{5}\right)^2$, where z_{mn} is the n^{th} positive zero of the Bessel function J_m.

c. What is the orthogonality relation is satisfied by the eigenfunctions? 14 pts

$\int_0^5 f_j(r) f_k(r)\, r\, dr = 0$ if $j \neq k$.

10. A certain non-isotropic material has a heat conductivity in the y direction that is 4 times the heat conductivity in the x direction, so that the equation for heat conduction is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} \quad -\infty < x < \infty,\ 0 \le y < \infty$$ 27 pts

At equilibrium we measure the temperature distribution along the x-axis to be:

$$u(x,0) = e^{-5x^2}, \quad -\infty < x < \infty.$$

Find the equilibrium heat distribution in the half plane . $-\infty < x < \infty,\ 0 \le y < \infty$ $\left(\frac{\partial u}{\partial t} = 0\right)$

$$u(x,y) = \frac{2y}{\pi}\int_{-\infty}^{\infty} e^{-5\bar{x}^2} \frac{1}{4\left(x-\bar{x}\right)^2 + y^2}\, d\bar{x}$$

11. A uranium rod of length π is initially at $0°C$. The ends of the rod sit in ice baths at $0°C$. Nuclear fission adds heat to the rod in a uniform manner, so that the temperature of the rod, u(x,t), satisfies:

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} + 1, \quad u(0,t) = \mathrm{u}(\pi,t) = 0, \quad u(x,0) = 0$$

a. Find an equilibrium temperature distribution v(x) for this problem. 10 pts.

$$v(x) = \frac{\pi x - x^2}{6}$$

b. Use v(x) to transform the problem into one with a homogeneous PDE. 12 pts

New unknown w, in terms of u and v: w = u - v

PDE for w: $\frac{\partial w}{\partial t} = 3\frac{\partial^2 w}{\partial x^2}$

Initial Condition: $w(x,0) = -v(x) = \frac{x^2 - \pi x}{6}$

Boundary Conditions: $w(0,t) = w(\pi,t) = 0$

c. Solve the transformed problem in (b). Do not solve for the Fourier coefficients, just show the formula. Use the result to solve the original problem.

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-9n^2 t},\ B_n = \frac{2}{\pi}\int_0^{\pi} \frac{x^2 - \pi x}{6}\sin(nx)\, dx,\ u(x,t) = w(x,t) + \frac{\pi x - x^2}{6}$$

12. Consider a rectangular metal plate, with coordinates 0< x < 2, 0 < y < 3. The temperature distribution u(x,y,t) of the plate obeys:

$$\frac{\partial u}{\partial t} = 5\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \frac{\partial u}{\partial x}(x,y,t) = 0 \text{ if x = 0 or 2, u(x,y,t) = 0 if y = 0 or 3.}$$

a. Find the smallest eigenvalue λ_1 of the problem: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0,\ (x,y) \in \Omega,$

with boundary conditions $\frac{\partial u}{\partial x}(x,y) = 0$ if x = 0 or 2, u(x,y) = 0 if y = 0 or 3, where Ω is the rectangle 0< x < 2, 0 < y < 3. 15 pts

$$\lambda_1 = \frac{\pi^2}{9} \text{ corresponding to } u(x,y) = 1 * \sin\left(\frac{\pi y}{3}\right)$$

b. Find a solution to the heat equation above, corresponding to λ_1. What is the ratio $\left(\max_{(x,y)\in\Omega} u(x,y,0)\right) \Big/ \left(\max_{(x,y)\in\Omega} u(x,y,\ln 2)\right)$ for this solution? 15 pts

$u(x,y,t) = \sin\left(\frac{\pi y}{3}\right) e^{-\frac{5\pi^2 t}{9}}$. Ratio is $2^{\frac{5\pi^2}{9}}$.

13. Solve the circularly symmetric heat equation:

$$\frac{\partial u}{\partial t} = \frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),\ 0 < r < 3,\ 0 < t,\ u(r = 3, t) = 0, \text{ with the initial condition}$$

$u(r,0) = 3 - r,\ 0 < r \le 3.$

a. $$u(r,t) = \sum_{n=1}^{\infty} A_n T_n(t) \phi_n(r)$$

Identify the eigenfunctions $\phi_n(r)$, the eigenvalues λ_n, and the functions $T_n(t)$. 10 pts

$$\phi_n(r) = J_{0n}\left(\sqrt{\lambda_{0n}}\ r\right),\ \lambda_{0n} \text{ satisfy } J_{0n}\left(\sqrt{\lambda_{0n}}\ 3\right) = 0,\ T_n(t) = e^{-2\lambda_{0n} t}.$$

b. Give an equation for A_n in terms of ϕ_n. 10 pts

$$A_n = \int_0^3 (3-r)\phi_n(r)\, r\, dr \Big/ \int_0^3 \phi_n(r)^2\, r\, dr$$

c. What is the orthogonality relation between ϕ_n and ϕ_j? 15 pts

$$\int_0^3 \phi_n(r)\phi_j(r)\, r\, dr = 0 \text{ if n} \ne \text{j.}$$

14. a. Find a Green's function G(x,s) for the problem: $u''(x) = f(x),\ 0 < x < 1,\ u(0) = u'(1) = 0.$

$$G(x,s) = \begin{cases} -x & x < s \\ -s & x \geq s \end{cases}$$

15 pts

b. Express the solution to: $u''(x) = f(x),\ 0 < x < 1,\ u(0) = u'(1) = 0$
in terms of the Green's function found in part a. Verify that the solution solves the differential equation and boundary conditions. 15 pts

$$u(x) = \int_0^1 f(s)G(x,s)\,ds = \int_0^x f(s)(-s)\,ds + \int_x^1 f(s)(-x)\,ds$$

Then $u(0) = \int_0^1 f(s)(0)\,ds = 0$, $u'(x) = -xf(x) + xf(x) - \int_x^1 f(s)\,ds$, $u'(1) = -\int_1^1 f(s)\,ds = 0$,

and $u''(x) = -\dfrac{d}{dx}\int_x^1 f(s)\,ds = f(x)$.

15. Solve the diffusion equation with convection:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x},\quad -\infty < x < \infty,\quad u(x,0) = f(x)$$

Hint: Take the Fourier transform with respect to x. Use the convolution theorem and the shift theorem. 30 pts

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(\bar{x}) e^{-(x+ct-\bar{x})^2/4kt}\, d\bar{x}$$

16. A fuel line on a fuel-injected car engine is subject to vibrations from the engine. The displacement u(x,t) from its undisturbed configuration satisfies:

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = \sin(\omega t)\sin(x),\ 0 < x < \pi,\ u(0,t) = u(\pi,t) = 0.$$

Assume $u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0.$

a. If $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\phi_n(x)$, what are the eigenfunctions $\phi_n(x)$ and the eigenvalues λ_n?

$\phi_n(x) = \sin(nx),\quad \lambda_n = n^2$ 10 pts

b. What differential equations and initial conditions are satisfied by $B_n(t)$? 12 pts

$$B_n''(t)+4n^2B_n(t)=\begin{cases}\sin(\omega t) & n=1\\ 0 & n\neq 1\end{cases},\quad B_n(0)=B'(0)=0$$

c. Solve for $B_n(t)$. For what positive value of ω does the solution blow up? 14 pts

$$B_1(t)=\begin{cases}\dfrac{1}{4-\omega^2}\left(\sin(\omega t)-\dfrac{\omega}{2}\sin(2t)\right), & \omega\neq 2\\ \dfrac{1}{8}\left(\sin(2t)-2t\cos(2t)\right) & \omega=2\end{cases}$$

$B_n(t)=0,\ \ n\neq 1$. Blows up for ω = 2.