MATH 3363
THE EQUATIONS OF MOTION FOR A VIBRATING STRING

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1. Derive the linear wave equation

Consider a perfectly flexible elastic string with equilibrium length 1. A *configuration* of the string is any twice differentiable map of $[0,1]$ into $\mathbb{R}^3$. The configuration given by $\gamma(s) = Ls\mathbf{k}$, $0 \le s \le 1$ will be called the *reference configuration* for the string. The variable s that parameterizes the reference configuration is a "material coordinate" for the string, so that s identifies the point of the string that is located at $\gamma(s)$ in the reference configuration, or at $\mathbf{r}(s)$ for any other configuration $\mathbf{r}$.

For simplicity we assume that there are no body forces such as gravity. We consider a *motion* $\mathbf{r} : [0,1]\times[0,\ \infty) \to \mathbb{R}^3$, subject to boundary conditions $\mathbf{r}(0,t) = \mathbf{0}$, $\mathbf{r}(1,t) = L\mathbf{k}$. $\mathbf{r}(s,t)$ denotes the position of the material point s at time t.

We assume that in the reference configuration, the string has mass density per unit volume $\rho(s)$ and cross section $A(s)$, so that $\rho(s)A(s)$ is the mass density per unit length.

The length of the portion of the string $s_1 \le s \le s_2$ is

$$\int_{s_1}^{s_2} \left\| \frac{d\mathbf{r}}{ds}(s,t) \right\| ds.$$

The quantity $\nu(s,t) = \left\| \frac{d\mathbf{r}}{ds}(s,t) \right\|$ is the *local stretch* of the string at (s,t). The string is subject to a "contact force" $\mathbf{n}(s,t)$ (the tension force). The effect of the contact force is best described by the equations of motion ($\mathbf{F} = m\mathbf{a}$):

$$\frac{d}{dt}\int_a^b \rho(s)A(s)\frac{\partial \mathbf{r}}{\partial t}(s,t)ds = \int_a^b \rho(s)A(s)\frac{\partial^2 \mathbf{r}}{\partial t^2}(s,t)ds = \mathbf{n}(b,t) - \mathbf{n}(a,t).$$

Then:

$$\int_a^b \rho(s)A(s)\frac{\partial^2 \mathbf{r}}{\partial t^2}(s,t) - \frac{\partial \mathbf{n}}{\partial s}(s,t)ds = 0$$

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for any $0 \le a,\ b \le 1$ and $t > 0$. We conclude that the integrand must vanish for all $0 < s < 1$ and $t > 0$:

$$\rho(s)A(s)\frac{\partial^2 \mathbf{r}}{\partial t^2}(s,t) - \frac{\partial \mathbf{n}}{\partial s}(s,t) = 0.$$

We assumed that the string is perfectly flexible. This means that $\mathbf{n}(s,t)$ is always tangent to the string (i.e., is parallel to $\frac{\partial \mathbf{r}}{\partial s}$) because the string always bends instantaneously to align itelf tangent to $\mathbf{n}(s,t)$, so that $\mathbf{n}(s,t) = N(s,t)\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}(s,t)$. We also assumed that the string is elastic. This means that $N(s,t)$ can be expressed as a function of the local stretch: $N(s,t) = \hat{N}(\nu(s,t))$.

We now calculate $\frac{\partial}{\partial s}\left(\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}(s,t)\right)$. Note that $\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}(s,t)$ is the standard unit tangent vector $\mathbf{T}$ for the curve $s \to \mathbf{r}(s,t)$. Then

$$\frac{\partial}{\partial s}\left(\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}(s,t)\right) = \hat{N}'(\nu(s,t))\frac{\partial \nu(s,t)}{\partial s}\mathbf{T} + \hat{N}(\nu(s,t))\frac{d}{ds}\mathbf{T}.$$

We know from Calculus III that $\frac{d}{ds}\mathbf{T}$ is normal to the curve and in fact is $\kappa \mathbf{N}\,\|\mathbf{r}_s\|$, where κ is the curvature of the string at (s,t) and $\mathbf{N}$ is the principal (unit) normal to the curve. (In Calculus III, an s used in this context would be an arc length parameter. Our s is not arc length). Thus, even before we restrict our attention to small amplitude vibrations, the tension force breaks down very neatly into a longitudinal force (tangent to the string) and a transverse force (normal to the string). We can see that the longitudinal force results from variations in the local stretch $\big(\hat{N}'(\nu(s,t))\frac{\partial \nu(s,t)}{\partial s}\big)$, while the transverse force results from curvature in the string.

Now we will assume that the motion is a small amplitude vibration and derive a linear partial differential equation. This means that we let

$$\mathbf{r}(s,t) = \epsilon\big(x(s,t)\mathbf{i} + y(s,t)\mathbf{j}\big) + \left(Ls + \epsilon z(s,t)\right)\mathbf{k}$$

and evaluate the derivative with respect to ϵ of the resulting partial differential equation at $\epsilon = 0$. We start with

(1.1)
$$\begin{aligned}\rho(s)A(s)\epsilon\big(x_{tt}\mathbf{i} + y_{tt}\mathbf{j} + z_{tt}\mathbf{k}\big) &= \frac{\partial}{\partial s}\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|} \\ &= \hat{N}'(\nu(s,t))\frac{\partial \nu(s,t)}{\partial s}\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|} + \hat{N}(\nu(s,t))\frac{\mathbf{r}_{ss}\,\|\mathbf{r}_s\|^2 - \mathbf{r}_s\left(\mathbf{r}_s \cdot \mathbf{r}_{ss}\right)}{\|\mathbf{r}_s\|^3}.\end{aligned}$$

Now recall that $\nu(s,t) = \|\mathbf{r}_s(s,t)\| = \sqrt{\mathbf{r}_s \cdot \mathbf{r}_s}$, so that

$$\frac{d}{ds}\nu(s,t) = \frac{\mathbf{r}_s \cdot \mathbf{r}_{ss}}{\|\mathbf{r}_s\|}.$$

Now compute $\mathbf{r}_s = \epsilon\left(x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}\right) + L\mathbf{k}$, $\mathbf{r}_{ss} = \epsilon\left(x_{ss}\mathbf{i} + y_{ss}\mathbf{j} + z_{ss}\mathbf{k}\right)$, and $\mathbf{r}_s \cdot \mathbf{r}_{ss} = \epsilon^2\left(x_s x_{ss} + y_s y_{ss} + z_s z_{ss}\right) + \epsilon L z_{ss}$.

The derivative of the left hand side of (1.1) with respect to ϵ is

$$\rho(s)A(s)\big(x_{tt}\mathbf{i} + y_{tt}\mathbf{j} + z_{tt}\mathbf{k}\big).$$

Now we compute

$$\frac{d}{d\epsilon}\left(\hat{N}'(\nu(s,t))\frac{\mathbf{r}_s \cdot \mathbf{r}_{ss}}{\|\mathbf{r}_s\|}\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|} + \hat{N}(\nu(s,t))\frac{\mathbf{r}_{ss}\left\|\mathbf{r}_s\right\|^2 - \mathbf{r}_s\left(\mathbf{r}_s \cdot \mathbf{r}_{ss}\right)}{\left\|\mathbf{r}_s\right\|^3}\right)$$

We must use the product rule to compute this. Note that when $\epsilon = 0$, $\mathbf{r}_s = L\mathbf{k}$, $\mathbf{r}_{ss} = 0$ and

$$\frac{d\mathbf{r}_{ss}}{d\epsilon} = x_{ss}\mathbf{i} + y_{ss}\mathbf{j} + z_{ss}\mathbf{k} \qquad \frac{d}{d\epsilon}\left(\mathbf{r}_s \cdot \mathbf{r}_{ss}\right) = L z_{ss}.$$

Thus any term involving an undifferentiated $\mathbf{r}_{ss}$ is zero. We obtain:

$$\begin{aligned}
&\hat{N}'(\nu(s,t))\frac{L z_{ss} L\mathbf{k}}{L^2} + \hat{N}(\nu(s,t))\frac{\left(x_{ss}\mathbf{i} + y_{ss}\mathbf{j} + z_{ss}\mathbf{k}\right)L^2 - L\mathbf{k}Lz_{ss}}{L^3} \\
&\quad = \hat{N}'(L)z_{ss}\mathbf{k} + \hat{N}(L)\frac{\left(x_{ss}\mathbf{i} + y_{ss}\mathbf{j}\right)}{L}
\end{aligned}$$

Plugging this result into (1.1) we obtain three linear partial differential quations:

$$\begin{aligned}
\rho(s)A(s)x_{tt} &= \frac{\hat{N}(L)}{L}x_{ss} \\
\rho(s)A(s)y_{tt} &= \frac{\hat{N}(L)}{L}y_{ss} \\
\rho(s)A(s)z_{tt} &= \hat{N}'(L)z_{ss}
\end{aligned}$$

Note that the two equations for tranverse motion, involving x and y, are identical, while the equation for longitudinal motion is different.

2. Energy conservation

We consider the same perfectly elastic and perfectly flexible string and show that the nonlinear equations (1), with $\mathbf{n}(s,t) = \hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}$, together with conservative boundary conditions at $s = 0,\ L$ conserve total mechanical energy.

We start with

$$0 = \rho(s)A(s)\frac{\partial^2 \mathbf{r}}{\partial t^2} - \frac{\partial}{\partial s}\left(\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}\right)$$

Since $\hat{N} : \mathbb{R} \to \mathbb{R}$ is continuous, there is a C^1 function $G : \mathbb{R} \to \mathbb{R}$ such that $G'(u) = \hat{N}(u)$. $G(u)$ is the stored energy corresponding to a local stretch u. Then

$$\begin{aligned}
0 &= \int_0^L \mathbf{r}_t \cdot \left(\rho(s)A(s)\frac{\partial^2 \mathbf{r}}{\partial t^2}(s,t) - \frac{\partial}{\partial s}\left(\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}\right)\right)\, ds \\
&= \int_0^L \rho(s)A(s)\frac{\partial}{\partial t}\frac{1}{2}\|\mathbf{r}_t\|^2 + \mathbf{r}_{st} \cdot \left(\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}\right)\, ds + \mathit{bdry\ terms} \\
&= \frac{d}{dt}\int_0^L \rho(s)A(s)\frac{1}{2}\|\mathbf{r}_t\|^2 + G(\nu(s,t))\, ds.
\end{aligned}$$

The boundary terms are:

$$\mathbf{r}_t \cdot \left(\hat{N}(\nu(s,t))\frac{\mathbf{r}_s}{\|\mathbf{r}_s\|}\right)\Bigg|_{s=0}^{L} = 0,$$

if the boundary conditions have the form $\mathbf{r}(0) = \mathbf{r}(L) = 0$ or $\mathbf{r}_s(0) = \mathbf{r}_s(L) = 0$. Thus the total mechanical energy of the string is constant in time.

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