MATH 4335
NOTES ON ANALYSIS FOR PARTIAL DIFFERENTIAL EQUATIONS

DAVID H. WAGNER

1. Introduction: Series of functions

In the study of partial differential equations, we often deal with series of functions:

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \qquad a \leq x \leq b.$$

It is important to understand whether and in what way a particular series converges. Usually this involves measuring the distance between functions using a norm: $\|f - g\|$. There are many norms that mathematicians use, but the ones that we use most frequently in Math 4335 are:

(1) The L^2 norm, $\left(\int_a^b |f(x)|^2 \, dx\right)^{\frac{1}{2}}$.
(2) The "sup" or L^∞ norm, $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$. Here "sup" is short for "supremum" and really means the least upper bound, which is what we use when a function does not actually attain a maximum value. A sequence of functions that converges in the "sup" norm converges uniformly.

A *normed vector space* is a vector space V with a norm on V, $\|\|$. The properties of a norm are:

(1) The norm maps elements of V to the non-negative real numbers (even for a complex vector space).
(2) For any $v \in V$, and any $t \in \mathbb{R}$, $\|tv\| = |t| \, \|v\|$.
(3) For any $u, \ v \in V$, $\|u + v\| \leq \|u\| + \|v\|$ (the triangle inequality).
(4) $\|v\| = 0 \implies v = 0$.

Remark. Technically, the L^2 norm satisfies $\|f\|_2 = 0 \implies f(x) = 0$ for "almost all" x; a function like this is considered to be the same as the zero function.

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Definition 1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed vector space V is called a *Cauchy sequence* if for every $\epsilon > 0$ there is $N < \infty$ such that for all $n,\ m > N$, $\|x_n - x_m\| < \epsilon$.

Definition 2. A normed vector space V is said to be *complete* if every Cauchy sequence in V converges to some $x \in V$. A complete normed vector space is called a *Banach space.*

This notion is important because it says that every sequence that ought to converge actually has an element of V to which it converges. Any useful normed vector space is complete.

Here is a simple test for convergence that generalizes the notion of absolute convergence of infinite series of numbers:

Theorem 1.1. *Let V be a Banach space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in V. If*

$$\sum_{n=1}^{\infty} \|x_n\| = r < \infty,$$

then there is $x \in V$ such that

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} x_n - x \right\| = 0.$$

Proof. Let $S_N = \sum_{n=1}^{N} x_n$. For $M > N$,

$$\|S_M - S_N\| = \left\| \sum_{n=N+1}^{M} x_n \right\| \le \sum_{n=N+1}^{\infty} \|x_n\|$$

But $\sum_{n=N+1}^{\infty} \|x_n\| = r - \sum_{n=1}^{N} \|x_n\|$ and tends to 0 as $N \to \infty$. Thus the sequence S_N is Cauchy. Since V is complete, there is $x \in V$ such that $\|S_N - x\| \to 0$ as $N \to \infty$. □

Remark. When the norm in Theorem 1.1 is $\|\|_{\infty}$, the theorem is called the *Weierstrass M-test.*

Example 1. Consider the sine series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$ on $[0, 2\pi]$. Since $\left\|\frac{1}{n^2} \sin nx\right\|_{\infty} = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, this series converges uniformly by Theorem 1.1.

Example 2. Theorem 1.1 does not work so well for L^2 convergence. Now consider the sine series $\sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$ on $[0, 2\pi]$. When we apply the test with $\|\|_{\infty}$ we get $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and in fact this series does not converge uniformly. If we apply the test with $\|\|_2$

we get $\sum_{n=1}^{\infty} \frac{\pi}{n} = \infty$. However this series converges in $L^2[0,\pi]$, because if $S_N(x) = \sum_{n=1}^{n} \frac{1}{n}\sin(nx)$,

$$\begin{aligned}\|S_N - S_M\|_2 &= \left(\sum_{n=M+1}^{N} \frac{1}{n^2}\|\sin(nx)\|_2^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=M+1}^{N} \frac{1}{n^2}\frac{\pi}{2}\right)^{\frac{1}{2}} \le \left(\sum_{n=M+1}^{\infty} \frac{1}{n^2}\frac{\pi}{2}\right)^{\frac{1}{2}}\end{aligned}$$

by the Parseval identity. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$),

$$\left(\sum_{n=M+1}^{\infty} \frac{1}{n^2}\frac{\pi}{2}\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}\left(\frac{\pi^2}{6} - \sum_{n=1}^{M} \frac{1}{n^2}\right)^{\frac{1}{2}} \to 0 \text{ as } M \to \infty.$$

Thus, as in the proof of Theorem 1.1, the partial sums $S_N(x)$ are a Cauchy sequence in $L^2[0,\pi]$ and hence converge to a function in $L^2[0,2\pi]$.

We have seen that a piecewise continuous function f on $-L < x < L$ can be represented by a Fourier series:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

where the equals sign holds when f is continuous at x, and does not hold when f is discontinuous at x. Using the L^2 inner product of functions on $[-L, L]$:

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)\, dx,$$

we find that the set of functions $\left\{1,\ \cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right),\ n = 1,\ 2,\ 3,\ \ldots\right\}$ is orthogonal. Consequently we can use the standard formula for the orthogonal projection of one vector onto another to determine the coefficients A_n and B_n:

$$A_n = \frac{\langle f, \cos\left(\frac{n\pi x}{L}\right)\rangle}{\left\|\cos\left(\frac{n\pi x}{L}\right)\right\|_2^2}, \quad B_n = \frac{\langle f, \sin\left(\frac{n\pi x}{L}\right)\rangle}{\left\|\sin\left(\frac{n\pi x}{L}\right)\right\|_2^2}.$$

Here the norm of a function is the L^2 norm on $[-L, L]$:

$$\|f\|_2^2 = \langle f, f\rangle = \int_{-L}^{L} |f(x)|^2\, dx.$$

It is natural to develop the theory of Fourier series in terms of the L^2 inner product and norm. The expansion of f as an infinite series of orthogonal functions is like writing the hypotenuse of a right triangle as a vector sum of the perpendicular sides– the only difference is that a Fourier series has infinitely many orthogonal pieces. For example, the Pythagorean theorem $c^2 = a^2 + b^2$ generalizes to the *Parseval identity*:

$$\|f\|_2^2 = \frac{1}{4}\|A_0\|_2^2 + \sum_{n=1}^{\infty}\left(\left\|A_n\cos\left(\frac{n\pi x}{L}\right)\right\|_2^2 + \left\|B_n\sin\left(\frac{n\pi x}{L}\right)\right\|_2^2\right) \tag{1.1}$$

$$= \frac{L}{2}A_0^2 + L\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right). \tag{1.2}$$

Naturally, if we only use finitely many of the orthogonal pieces, we get an inequality, called *Bessel's inequality*:

$$\frac{L}{2}A_0^2 + L\sum_{n=1}^{N}\left(A_n^2 + B_n^2\right) \leq \|f\|_2^2.$$

Now I want to discuss how derivatives work with Fourier series. First, we need to think about derivatives in terms of functions. We say that g is the derivative of f, if

$$f(x) - f(0) = \int_0^x g(t)\,dt \quad \text{and} \quad \int_{-L}^{L}|g(t)|\,dt < \infty. \tag{1.3}$$

We say that f has an L^2 derivative if (1.3) holds and $\|g\|_2 < \infty$. Using this notion, we can say that the derivative of the absolute value function is the step function

$$g(x) = \begin{cases} -1 & -L < x < 0 \\ 1 & 0 < x < L. \end{cases}$$

Note that the value of $g(0)$ does not matter, and in fact the absolute value function has no derivative at 0.

A function f that has a derivative function g as in (1.3) is said to be *absolutely continuous*.

A function f for which $\|f\|_2 < \infty$ is said to be in $L^2[-L, L]$. If $f \in L^2[-L, L]$ and $f' \in L^2[-L, L]$, we say that $f \in H^1[-L, L]$. H^1 is called a *Sobolev space*, and much of the theory of partial differential equations is done in the context of one or more Sobolev spaces.

Theorem 1.2. *If $f \in H^1[-L, L]$, then for all $x,\ y \in [-L, L]$,*

$$|f(y) - f(x)| \leq \|f'\|_2\sqrt{|y - x|}.$$

Proof.

$$\begin{aligned}|f(y)-f(x)| &= \left|\int_x^y f'(s)ds\right| \\ &= \left|\int_x^y 1\ f'(s)ds\right| \le \left|\int_x^y 1^2 ds\right|^{\frac{1}{2}} \|f'\|_2 = \|f'\|_2 \sqrt{|y-x|}.\end{aligned}$$

by the Schwartz inequality. □

The next theorem is a special case of the *Sobolev embedding theorem.*

Theorem 1.3. *If $f \in H^1[-L,L]$ and f has period $2L$, then:*

(1) *The derivative function of f, f', has a Fourier series which converges to f' in $L^2[-L,L]$.*
(2) *The Fourier series for f converges uniformly to f.*
(3) *The Fourier series for f' can be computed by differentiating the Fourier series for f term-by-term.*

Proof. The first statement follows from the assumption that f' is in L^2. For the second statement, first let us calculate the Fourier coefficients of f'. We seek to find coefficients a_0 and $a_n,\ b_n,\ n = 1,\ 2, 3, \ldots$, such that

$$f'(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Since f has period $2L$,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f'(x)\ dx = \frac{1}{L}\left(f(L) - f(-L)\right) = 0.$$

Note how this has used (1.3). Next,

$$a_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \tag{1.4}$$

$$= \frac{1}{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=-L}^{L} - \frac{1}{L}\frac{n\pi}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \tag{1.5}$$

$$= 0 - \frac{n\pi}{L}B_n. \tag{1.6}$$

Thus $a_n = -\frac{n\pi}{L}B_n$, and similarly $b_n = \frac{n\pi}{L}A_n$. This is exactly what we obtain from differentiating the series for f term-by-term. Now to prove the second statement. The key to making this valid is the convergence of the resulting series to f'. However, as noted before, when $f' \in L^2[-L, L]$ this convergence is guaranteed by the L^2 (mean-square) convergence of Fourier series for functions in L^2. Thus, when we apply Parseval's identity to f', we get:

$$\begin{aligned} \|f'\|_2^2 &= \sum_{n=1}^{\infty} \left(\left\| a_n \cos\left(\frac{n\pi x}{L}\right) \right\|_2^2 + \left\| b_n \sin\left(\frac{n\pi x}{L}\right) \right\|_2^2 \right) \\ &= \sum_{n=1}^{\infty} (n\pi)^2 \left(A_n^2 + B_n^2\right) \end{aligned}$$

Thus, by the Schwartz inequatliy,

$$\begin{aligned} \sum_{n=1}^{\infty} |A_n| + |B_n| &= \sum_{n=1}^{\infty} \frac{1}{n} n \left(|A_n| + |B_n|\right) \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} m^2 \left(A_m^2 + B_m^2\right) \right)^{\frac{1}{2}} < \infty \end{aligned}$$

Consequently, the Fourier series for f converges uniformly. □

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, 3550 CULLEN BLVD., HOUSTON, TX 77204-3008

URL: `http://www.math.uh.edu/~wagner`