No segment of the form
\[ \left( \frac{3k + 1}{3^m}, \frac{3k + 2}{3^m} \right), \]
where \( k \) and \( m \) are positive integers, has a point in common with \( P \). Since every segment \((a, b)\) contains a segment of the form (24), if
\[ 3^{-m} < \frac{b - a}{6}, \]
P contains no segment.

To show that \( P \) is perfect, it is enough to show that \( P \) contains no isolated point. Let \( x \in P \), and let \( S \) be any segment containing \( x \). Let \( I_n \) be that interval of \( E_n \) which contains \( x \). Choose \( n \) large enough, so that \( I_n \subset S \). Let \( x_n \) be an endpoint of \( I_n \), such that \( x_n \neq x \).

It follows from the construction of \( P \) that \( x_n \in P \). Hence \( x \) is a limit point of \( P \), and \( P \) is perfect.

One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero (the concept of measure will be discussed in Chap. 11).

**CONNECTED SETS**

2.45 Definition Two subsets \( A \) and \( B \) of a metric space \( X \) are said to be separated if both \( A \cap B \) and \( A \cap B \) are empty, i.e., if no point of \( A \) lies in the closure of \( B \) and no point of \( B \) lies in the closure of \( A \).

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval \([0, 1]\) and the segment \((1, 2)\) are not separated, since 1 is a limit point of \((1, 2)\). However, the segments \((0, 1)\) and \((1, 2)\) are separated.

The connected subsets of the line have a particularly simple structure:

2.47 Theorem A subset \( E \) of the real line \( \mathbb{R}^1 \) is connected if and only if it has the following property: If \( x \in E \), \( y \in E \), and \( x < z < y \), then \( z \in E \).

Proof If there exist \( x \in E \), \( y \in E \), and some \( z \in (x, y) \) such that \( z \notin E \), then \( E = A_y \cup B_y \) where
\[ A_y = E \cap (-\infty, z), \quad B_y = E \cap (z, \infty), \]
Since \( x \in A_y \) and \( y \in B_y \), \( A \) and \( B \) are nonempty. Since \( A_y \subset (-\infty, z) \) and \( B_y \subset (z, \infty) \), they are separated. Hence \( E \) is not connected.

To prove the converse, suppose \( E \) is not connected. Then there are nonempty separated sets \( A \) and \( B \) such that \( A \cup B = E \). Pick \( x \in A \), \( y \in B \), and assume (without loss of generality) that \( x < y \). Define
\[ z = \sup (A \cap [x, y]). \]
By Theorem 2.28, \( z \notin A \); hence \( z \notin B \). In particular, \( x \leq z < y \).

If \( z \notin A \), it follows that \( x < z < y \) and \( z \notin E \).

If \( z \in A \), then \( z \notin B \), hence there exists \( z_1 \) such that \( z < z_1 < y \) and \( z_1 \notin B \). Then \( x < z_1 < y \) and \( z_1 \notin E \).

**EXERCISES**

1. Prove that the empty set is a subset of every set.
2. A complex number \( z \) is said to be algebraic if there are integers \( a_0, \ldots, a_n \), not all zero, such that
\[ a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0. \]

Prove that the set of all algebraic numbers is countable. **Hint:** For every positive integer \( N \) there are only finitely many equations with
\[ n + \mid a_0 \mid + \mid a_1 \mid + \cdots + \mid a_n \mid = N. \]

3. Prove that there exist real numbers which are not algebraic.
4. Is the set of all irrational real numbers countable?
5. Construct a bounded set of real numbers with exactly three limit points.
6. Let \( E \) be the set of all limit points of a set \( E \). Prove that \( E \) is closed. Prove that \( E \) and \( E \) have the same limit points. (Recall that \( \bar{E} = E \cup E \)). Do \( E \) and \( E \) always have the same limit points?
7. Let \( A_1, A_2, A_3, \ldots \) be subsets of a metric space.
(a) If \( B_n = \bigcup_{i=1}^n A_i \), prove that \( B_n = \bigcup_{i=1}^n A_i \), for \( n = 1, 2, 3, \ldots \).
(b) If \( B_n = \bigcup_{i=1}^n A_i \), prove that \( B_n = \bigcup_{i=1}^n A_i \).

Show, by an example, that this inclusion can be proper.
8. Is every point of every open set \( E \subset \mathbb{R} \) a limit point of \( E \)? Answer the same question for closed sets in \( \mathbb{R} \).
9. Let \( E^c \) denote the set of all interior points of a set \( E \). [See Definition 2.18(e); \( E^c \) is called the **interior of \( E \)**.]
(a) Prove that \( E^c \) is always open.
(b) Prove that \( E^c \) is open if and only if \( E \) is open.
(c) If \( G \subset E \) and \( G^c \) is open, prove that \( G \subset E \).
(d) Prove that the complement of \( E \) is the closure of the complement of \( E \).
(e) Do \( E \) and \( E \) always have the same interiors?
(f) Do \( E \) and \( E \) always have the same closures?