MATH 3339 - 03 15951
Statistics for the Sciences
Discrete Random Variables

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Lecture 14 - 3339
1. Expected Values and Variance of Discrete Variables

2. Bernoulli Random Variables

3. Binomial Random Variables
Popper Set Up

- Fill in all of the proper bubbles.
- Make sure your ID number is correct.
- Make sure the filled in circles are very dark.
- This is popper number 09.
Twelve babies spoke for the first time at the following ages (in months):

8  9  10  11  12  13  15  15  18  20  20  26

1. Using the 1.5 IQR rule, give the boundaries of the outliers.
   a) (8, 26)
   b) (10.5, 19)
   c) (2, 27.5)
   d) (-2.25, 31.75)
   e) None of these
2. If we plotted the heights of the team members on the cartoon, the graph would be

- skewed right
- skewed left
- roughly symmetrical
- there is no way to tell without more data
Suppose the class average (mean) of test 1 is 60 with a standard deviation of 6. I decide to curve by multiplying each test score by 1.5 then subtracting 10.

3. The new class average (mean) is
   a. 70
   b. 75
   c. 80
   d. 85

   The new class average (mean) calculation is:
   \[ Y = 1.5X - 10 \]
   \[ = 1.5 \times 60 - 10 \]
   \[ = 90 - 10 = 80 \]

4. The new class standard deviation is
   a. 9
   b. 13.5
   c. 3.5
   d. 8.4

   The new class standard deviation calculation is:
   \[ 1.5 \sigma \]
   \[ = 1.5 \times 6 = 9 \]
Expected Value and Variance

The **expected value** or mean of the distribution of a random variable $X$ is given by:

$$E[X] = \mu = \sum x \cdot f(x) = \sum_{i=1}^{n} x_i \cdot p_i$$

The **variance** of a rv $X$ is

$$\sigma^2 = Var[X] = E[(X - \mu)^2] = E[X^2] - E[X]^2$$
Properties of Expected Values and Variance

- \( E[c] = c \) for any constant \( c \in \mathbb{R} \)
- \( E[aX + b] = aE[X] + b \)
- \( E[aX + bY] = aE[X] + bE[Y] \)
- \( E[h(x)] = \sum_x h(x) \cdot f(x) \)
- \( \text{Var}[aX + b] = a^2 \text{Var}[X] \)
- \( \text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] \)

\[ \text{SD}(aX + bY) = a \text{SD}(X) + b \text{SD}(Y) \] \( \text{No!} \)

\[ \text{SD}(aX + bY) = \sqrt{\text{Var}(aX + bY)} = \sqrt{a^2 \text{Var}(X) + b^2 \text{Var}(Y)} \] \( \text{Yes!} \)
Expected Values and Variance

Example: Let $X$ have pmf. given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.4</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Determine $E[X]$, $E[X^2]$, $Var[X]$ and the standard deviation of $X$.

\[ E(X) = 1 \cdot (0.4) + 2 \cdot (0.2) + 3 \cdot (0.3) + 4 \cdot (0.1) = 2.1 \]

\[ E(X^2) = 1^2 \cdot (0.4) + 2^2 \cdot (0.2) + 3^2 \cdot (0.3) + 4^2 \cdot (0.1) = 5.5 \]

\[ Var(X) = E(X^2) - [E(X)]^2 = 5.5 - 2.1^2 = 1.09 \]

\[ SD(X) = \sqrt{Var(X)} = \sqrt{1.09} = 1.044 \]
Expected Values and Variance

Example: Determine the expected value and variance of the rv $Y$ defined by $Y = 5X - 1$, where $X$ is given in the previous problem.

$$E(Y) = E(5X - 1) = 5E(X) - 1 = 5(2.1) - 1 = 9.5$$

$$Var(Y) = Var(5X - 1) = 5^2 \cdot Var(X)$$

$$= 25 \times 1.09$$

$$= 27.25$$
Expected Values and Variance

Example: A store carries flash drives with either 1 GB, 2 GB, 4 GB, 8 GB, or 16 GB of memory. The table below gives the distribution of the random variable $X = \text{the amount of memory in a purchased drive}$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>0.05</td>
<td>0.10</td>
<td>0.35</td>
<td>0.40</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Determine $E[X]$ and $\text{Var}[X]$ using R.

```r
> X=c(1,2,4,8,16)
> PX=c(0.05,0.1,0.35,0.4,0.1)
> EX=sum(X*PX)
> EX
[1] 6.45
> EX2=sum(X^2*PX)
> EX2
[1] 57.25
> VarX= EX2-EX^2
> VarX
[1] 64.75
```
Finding Expected Value and Standard Deviation in R

```r
> X=c(1,2,4,8,16)
> PX=c(0.05,0.1,0.35,0.4,0.1)
> EX=sum(X*PX)
> EX
[1] 6.45
> EX2=sum(X^2*PX)
> EX2
[1] 57.25
> VarX= EX2-EX^2
> VarX
[1] 15.6475
```
Special cases for Discrete Random Variables:

- Bernoulli Random Variable
- Binomial Random Variable
- Hypergeometric Distribution
- Poisson Distribution
- Jointly Distribution Variables
The Bernoulli Probability Distribution

Definition: A **Bernoulli** random variable is a random experiment (a Bernoulli Trial) with the following characteristics:

1. The outcome can be classified as either **success** or **failure** (where these are mutually exclusive and exhaustive).

2. The probability of **success** is $p$, so the probability of **failure** is $q=1-p$. e.g. a coin is flipped (heads or tails), someone is pulled over for speeding (ticket or warning), etc.

Suppose that a coin is flipped. Let $X$ be the random variable that indicates that heads was flipped. Here heads represents “success" and tails represents “failure" so that $X$ is a Bernoulli random variable.
Probability Function for Bernoulli Variable

possible values: 0, 1

\[ f(x) = P(X = x) = \begin{cases} 
  p, & \text{if } x = 1 \\
  1 - p, & \text{if } x = 0 \\
  0, & \text{if } x \neq 0, 1 
\end{cases} \]

A compact way of writing this is:

\[ f(x) = P(X = x) = p^x (1 - p)^{1-x} \]

\[ \uparrow \]

\[ x \text{ can be } 0, 1 \]

\[ P(X = 0) = p^0 (1 - p)^{1-0} = 1 - p \]

\[ P(X = 1) = p^1 (1 - p)^{1-1} = p \]
Mean and Variance of a Bernoulli Distribution

If $X$ has the Bernoulli distribution with probability of success $p$, the mean and variance of $X$ are

$$
\mu_X = E[X] = p
$$

$$
\sigma^2_X = Var[X] = p(1 - p)
$$

Then the standard deviation is the square root of the variance.

$$
sd(X) = \sqrt{Var(X)} = \sqrt{p(1-p)}
$$
The Binomial Probability Distribution

Suppose that we flip a coin 10 times. This is a sequence of Bernoulli trials. We are interested in calculating the probability of obtaining a certain number of heads. Let \( X_i \) indicate heads on the \( i \)-th flip.

Define \( Y = X_1 + X_2 + \ldots + X_{10} \). What does \( Y \) represent?

Among these 10 flips, how many "Head" will we get?

What is the probability that \( Y = 0 \)?

\( Y \) can be 0, 1, 2, ..., 10

What is the probability that \( Y = 1 \)?
The Binomial Probability Distribution

Suppose that we flip a coin 10 times. This is a sequence of Bernoulli trials. We are interested in calculating the probability of obtaining a certain number of heads. Let $X_i$ indicate heads on the $i$-th flip.

Define $Y = X_1 + X_2 + \ldots + X_{10}$. What does $Y$ represent?

What is the probability that $Y = 2$?

What is the probability that $Y = n$?
The Binomial Probability Distribution

Here $Y$ is the sum of 10 independent Bernoulli trials. We call this type of random variable a **Binomial random variable**.

A random variable $X$ is a Binomial random variable if the following conditions are satisfied:

1. $X$ represents the number of successes on $n$ Bernoulli trials.
2. The probability of success for each trial is $p$.
3. The trials are mutually independent.

If $X$ is a binomial random variable with probability $p$ of success on each of $n$ trials, we write $X \sim Binomial(n, p)$

If $X \sim Binomial(n, p)$, then $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$ where $x = 0, 1, 2, \ldots, n$
If a count $X$ has the Binomial distribution with number of observations $n$ and probability of success $p$, the mean and variance of $X$ are

\[
\mu_X = E[X] = np
\]
\[
\sigma^2_X = Var[X] = np(1 - p)
\]

Then the standard deviation is the square root of the variance.
The Binomial Probability Distribution

R commands:
The Binomial Probability Distribution

Example: Suppose that at a 4-way stop in a certain subdivision, only 12% of drivers come to a complete stop. What is the probability that among 8 drivers, at least 6 of them will run the stop sign?

What is the expected number of drivers who will run the stop sign?
Suppose $X \sim \text{Binomial}(12, 0.3)$, find the following:

$P(2 \leq X < 5) =

P(X > 5) =$
Example

Suppose it is known that 80% of the people exposed to the flu virus will contract the flu. Out of a family of five exposed to the virus, what is the probability that:

1. No one will contract the flu?

2. All will contract the flu?

3. Exactly two will get the flu?

4. At least two will get the flu?