MATH 3339 - 03 15951
Statistics for the Sciences
Continuous Random Variables

Wendy Wang
wwang60@central.uh.edu

Lecture 19 - 3339
Outline

1. Continuous Random Variables
2. Probability Density Function
3. Uniform Distribution
Popper Set Up

- Fill in all of the proper bubbles.
- Make sure your ID number is correct.
- Make sure the filled in circles are very dark.
- This is popper number 15.

*choose d for 11-15*
Types of Random Variables

- A random variable that may assume either a finite number of values or an infinite sequence of values such as 0, 1, ... is referred to as a **discrete random variable**.

- A random variable that may assume any numerical value in an interval or collection of intervals is called a **continuous random variable**.
Example

Suppose we want to determine the probability of waiting for an elevator where the longest waiting time is 5 minutes.

What type of variable do we have?

Suppose we take a sample of 10, 50, 1000, and 10,000 people to see how long they wait for the elevator. The following are histograms for the waiting times of each sample.
Sample of 10 People Waiting for the Elevator

Histogram of a Sample of 10

$0.4 \times 10 = 4$

$0.3 \times 10 = 3$

$2$

$1$
Sample of 50 People Waiting for the Elevator

Histogram of a Sample of 50
Sample of 100 People Waiting for the Elevator

Histogram of a Sample of 100

Density

0.00 0.05 0.10 0.15 0.20 0.25

0 1 2 3 4 5
Sample of 1000 People Waiting for the Elevator

Histogram of a Sample of 1000 Minutes

Density

0 1 2 3 4 5

0.00 0.05 0.10 0.15 0.20 0.25

Wendy Wang wwang60@central.uh.edu MATH 3339 Lecture 19 - 3339 9/24
Sample of 10,000 People Waiting for the Elevator

Area = base \times height

Histogram of a Sample of 10,000

\[ l = 5 \cdot \frac{1}{5} \]
A **probability distribution** for random variables describes how probabilities are distributed over the values of the random variable.

For a discrete random variable \( X \), the probability distribution is defined by **probability mass function**, denoted by \( f(x) \). This provides the probability for each value of the random variable.

For a continuous random variable, this is called the **probability density function** \( f(x) \). The probability density function (pdf) \( f(x) \) is a graph of an equation. The area under the graph of \( f(x) \) corresponding to a given interval provides the probability that the random variable \( X \) assumes a value in that interval.
discrete r.v.: \[ \text{pmf: } f(x) = P(X = x) \]

continuous r.v.: \[ \text{pdf: } f(x) \neq P(X = x) \]
\[ P(X = x) = 0 \text{ for all } x \]

for pdf: 
\[ P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx \]
\[ P(x = a) = P(a \leq X \leq a) = 0 \]
Probability Density Function

For $f(x)$ to be a legitimate pdf, it must satisfy the following two conditions:

1. $f(x) \geq 0$ for all $x$.

2. The area under the entire graph of $f(x)$ must equal 1.

$$\int_{-\infty}^{\infty} f(x) \, dx = 1$$

$$P(x \leq x) = \int_{-\infty}^{x} f(t) \, dt = F(x)$$
Density Curve for Elevator Waiting Times

\[ \text{Area} = \text{base} \cdot \text{height} = 5 \cdot (0.2) = 1 \]

\[ P(X = 4) = 0 \]

\[ P(3 \leq X \leq 5) = (5 - 3) \cdot 0.2 = 0.4 \]

\[ P(3 \leq X \leq 3.5) = 0.5 \cdot (0.2) = 0.1 \]
Uniform Distribution

A continuous random variable $X$ is said to have a uniform distribution on the interval $[A, B]$ if the pdf of $X$ is:

$$f(x) = \begin{cases} \frac{1}{B-A}, & A \leq x \leq B \\ 0, & \text{otherwise} \end{cases}$$

**Example:** $X$ is defined on $[0, 5]$

$$f(x) = \begin{cases} \frac{1}{5}, & 0 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{5} f(x) \, dx + \int_{5}^{\infty} f(x) \, dx$$
Density curve for waiting time

The rectangle ranges between 0 and 5. The height of the rectangle is:

\[
\frac{1}{\text{highest value} - \text{lowest value}} = \frac{1}{5 - 0} = 0.2.
\]
The probability of any event between a range of values is the same as the area between the range under the density curve.

Area of rectangle = height × width = 0.2 × 2 = 0.4
Example continued

What is the probability that a person waits for at least one minute?
What is the probability that a person waits for at least one minute?

\[ P(X \geq 1) = \text{area above } 1 \]
\[ = \text{height } \times \text{width} \]
Example

Consider a spinner that, after a spin, will point to a number between zero and 1 with “uniform probability.”

1. What is the probability that the spinner will land on something less than 0.75?

\[ P(X < 0.75) = 0.75 \cdot 1 = 0.75 \]

\[ \Leftrightarrow P(X \leq 0.75) \]

2. Determine \( P\left(\frac{1}{5} \leq X \leq \frac{3}{8}\right) \).

\[ P\left(\frac{1}{5} \leq X \leq \frac{3}{8}\right) = \left(\frac{3}{8} - \frac{1}{5}\right) \cdot 1 = \frac{7}{40} \]

3. Determine the value of \( x_0 \) such that \( P(X \leq x_0) = 0.5 \)

\[ X_0 = 0.5 \]

4. Determine the value of \( X_0 \) such that \( P(X \geq X_0) = 0.35 \)

\[ X_0 = 0.65 \]
Example of a density function

Let the random variable $X$ = a dealer’s profit, in units of $5000$, on a new automobile with a density function:

$$f(x) = \begin{cases} 
2(1 - x) & \text{for } 0 < x < 1 \\
0 & \text{elsewhere}
\end{cases}$$

What is the probability that the dealer’s profit is at least $4000$ for a new automobile. That is $P(X \geq \frac{4000}{5000}) = P(X \geq 0.8)$.

$$P(X \geq 0.8) = \int_{0.8}^{1} 2(1-x) \, dx$$

$$= \left[ 2x - x^2 \right]_{0.8}^{1}$$

$$= 2(1 - 0.8) - (1^2 - 0.8^2) = \frac{1}{25}$$
Finding Probability

To find the probability of the profit at least $4000, we need to find the area under the curve between 0.8 and 1.
Density Function

This is the graph of the density function.

\[ f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & 0 \leq x \leq 0.8 \end{cases} \]

\[ P(x \geq 0.8) \]

Area = \( \frac{1}{2} \cdot \text{base} \cdot \text{height} \)

= \( \frac{1}{2} \cdot (1 - 0.8) \cdot 0.4 \)

(when \( x = 0.8 \), \( f(x) = 2(1 - 0.8) \))

= \( \frac{1}{25} \)
Definition of a Density Function

- **A density function** is a nonnegative function $f$ defined of the set of real numbers such that:

$$
\int_{-\infty}^{\infty} f(x) \, dx = 1.
$$

- If $f$ is a density function, then its integral $F(x) = \int_{-\infty}^{x} f(u) \, du$ is a continuous cumulative distribution function (cdf), that is $P(X \leq x) = F(x)$.

- If $X$ is a random variable with this density function, then for any two real numbers, $a$ and $b$

$$
P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx.
$$