MATH 3339 - 03 15951
Statistics for the Sciences
Continuous Random Variables

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Lecture 20 - 3339
Outline

1. Cumulative Distribution Functions

2. Expected Values
Popper Set Up

- Fill in all of the proper bubbles.
- Make sure your ID number is correct.
- Make sure the filled in circles are very dark.
- This is popper number 16.
Definition of a Density Function

- A **density function** is a nonnegative function $f$ defined of the set of real numbers such that:

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1.
\]

- If $f$ is a density function, then its integral $F(x) = \int_{-\infty}^{x} f(u) \, du$ is a continuous cumulative distribution function (cdf), that is $P(X \leq x) = F(x)$.

- If $X$ is a random variable with this density function, then for any two real numbers, $a$ and $b$

\[
P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx.
\]
"Time headway" in traffic flow is the elapsed time between the time that one car finishes passing a point and the instant that the next car begins to pass that point. Let $X$ = the time headway (in sec) for two randomly chosen consecutive cars on a freeway during a period of heavy flow. The following pdf of $X$ is essentially the one suggested in "The Statistical Properties of Freeway Traffic" (Transp. Res., vol. 11: 221 - 228):

$$f(x) = \begin{cases} 
0.15e^{-0.15(x-0.5)} & x \geq 0.5 \\
0 & \text{otherwise}
\end{cases}$$
Density Function

This is the graph of the density function.

Density Curve for Headway Time

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Determine Probability

What is the probability that headway time is at most 5 seconds.
Cumulative Distribution Function Properties

Any cumulative distribution function $F$ has the following properties:

1. $F$ is a nondecreasing function defined on the set of all real numbers.

2. $F$ is right-continuous. That is, for each $a$,
   \[ F(a) = F(a+) = \lim_{x \to a^+} F(x). \]

3. $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to +\infty} F(X) = 1$.

4. $P(a < X \leq b) = F_x(b) − F_x(a)$ for all real $a$ and $b$, $a < b$.

5. $P(X > a) = 1 − F_x(a)$.

6. $P(X < b) = F_x(b−) = \lim_{x \to b−} F_x(x)$.

7. $P(a < X < b) = F_x(b−) − F_x(a)$.

8. $P(X = b) = F_x(b) − F_x(b−)$. 
A continuous random variable $X$ is said to have a uniform distribution on the interval $[A, B]$ if the pdf of $X$ is:

$$f(x) = \begin{cases} \frac{1}{B-A}, & A \leq x \leq B \\ 0, & \text{otherwise} \end{cases}$$
Determine the cdf of a Uniform Distribution
Cumulative Density Function

Cumulative Density Curve for Elevator Waiting Times
Using the cdf $F(X)$ to Compute Probabilities

Let $X$ be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. Then for any number $a$, 

$$P(X > a) = 1 - F(a)$$

and for any two numbers $a$ and $b$ with $a < b$, 

$$P(a \leq X \leq b) = F(b) - F(a)$$
Example

The cdf for $X =$ measurement error is

$$F(x) = \begin{cases} 
0 & x < -2 \\
\frac{1}{2} + \frac{3}{32} \left(4x - \frac{x^3}{3}\right) & -2 \leq x < 2 \\
1 & 2 \leq x
\end{cases}$$

1. Compute $P(X < 0)$.

2. Compute $P(-1 < X < 1)$

3. Compute $P(X > 0.5)$
Going from CDF to PDF

The cdf for $X =$ measurement error is

$$F(x) = \begin{cases} 
0 & x < -2 \\
\frac{1}{2} + \frac{3}{32} \left( 4x - \frac{x^3}{3} \right) & -2 \leq x < 2 \\
1 & 2 \leq x 
\end{cases}$$

Determine the PDF $f(x)$. 
Example

Suppose we have a pdf of

\[ f(x) = \begin{cases} \frac{3}{8} x^2 & 0 \leq X \leq k \\ 0 & \text{otherwise} \end{cases} \]

a) Determine \( k \).

b) Give the cdf of this distribution.

c) Determine \( x_0 \) such that \( P(X \leq x_0) = 0.125 \)
Quantiles

Let $F$ be a given cumulative distribution and let $p$ be any real number between 0 and 1. The \textbf{(100$p$)th percentile} of the distribution of a continuous random variable $X$ is defined as

$$F^{-1}(p) = \min\{x \mid F(x) \geq p\}.$$ 

For continuous distributions, $F^{-1}(p)$ is the smallest number $x$ such that $F(x) = p$. 
Determine the Percentiles

Given a cdf,

\[ F(x) = \begin{cases} 
0 & X < 0 \\
\frac{1}{8}x^3 & 0 \leq X \leq 2 \\
1 & X > 2 
\end{cases} \]

1. Determine the 90\(^{th}\) percentile.

2. Determine the 50\(^{th}\) percentile.

3. Find the value of \(c\) such that \(P(X \leq c) = 0.75\).
The **expected** or **mean value** of a continuous random variable $X$ with pdf $f(x)$ is

$$E(X) = \int_{-\infty}^{\infty} xf(x) \, dx.$$  

More generally, if $h$ is a function defined on the range of $X$,

$$E(h(X)) = \int_{-\infty}^{\infty} h(x)f(x) \, dx.$$
Example

The following is a pdf of $X$,

$$f(x) = \begin{cases} \frac{3}{2}(1 - x^2) & 0 \leq X \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Determine $E(X)$.

2. Determine $E(X^2)$
Mean and Variance of the Uniform Distribution

Let $X \sim \text{Unif}(a, b)$

- $E(X) = \frac{a+b}{2}$

- $\text{Var}(X) = \frac{(b-a)^2}{12}$
Example From Quiz 8

Let $X$ be the amount of time (in hours) the wait is to get a table at a restaurant. Suppose the cdf is represented by

$$F(X) = \begin{cases} 
0 & x < 0 \\
\frac{x^2}{9} & 0 \leq x \leq 3 \\
1 & x > 3 
\end{cases}$$

Use the cdf to determine $E[X]$. 