MATH 2331 - 19859

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Definition

A solution of a system of linear equations is a list of numbers $(s_1, s_2, ..., s_n)$ that makes each equation in the system true when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$, respectively.

There are three possibilities:

1. Exactly one solution exists.
2. No solution exists.
3. Infinitely many solutions exist.

Definition

The essential information of a linear system can be recorded in an array called matrix.
Equivalent Systems

STRATEGY FOR SOLVING A SYSTEM: Replace one system with an equivalent system that is easier to solve using three basic operations:

1. Replace one equation by the sum of itself and a multiple of another equation.
2. Interchange two equations.
3. Multiply all the terms in an equation by a nonzero constant.

Elementary Row Operations

1. Add one row to a multiple of another row.
2. Interchange two rows.
3. Multiply all entries in a row by a nonzero constant.
Fundamental Questions

1. Does a solution exist?  
   If so, the system is called *consistent*.

2. If a solution exists, is it *unique*?
Example

Is this system consistent?

\[
\begin{align*}
   x_1 & - 2x_2 + x_3 = 0 \\
   2x_2 & - 8x_3 = 8 \\
   -4x_1 & + 5x_2 + 9x_3 = -9
\end{align*}
\]

Last time, this system was reduced to the triangular form:

\[
\begin{align*}
   x_1 & - 2x_2 + x_3 = 0 \\
   x_2 & - 4x_3 = 4 \\
   x_3 & = 3
\end{align*}
\]

\[
\begin{bmatrix}
   1 & -2 & 1 & 0 \\
   0 & 1 & -4 & 4 \\
   0 & 0 & 1 & 3
\end{bmatrix}
\]

At this point we can already tell that the solution exists. Why?
Exercise

For what values of $h$ will the system be consistent?

\[
\begin{align*}
3x_1 - 9x_2 &= 12 \\
-2x_1 + 6x_2 &= h
\end{align*}
\]
Row reduction and echelon form

Section 1.2
Echelon Forms

Echelon form (or row echelon form)

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

(a) \[
\begin{bmatrix}
\color{black}1 & * & * & * & * \\
0 & \color{black}1 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
\color{black}1 & * & * \\
0 & 0 & \color{black}1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
0 & \color{black}1 & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & \color{black}1 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & \color{black}1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & \color{black}1 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & \color{black}1 & * & * & *
\end{bmatrix}
\]
Reduced echelon form

Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

\[
\begin{bmatrix}
0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
\end{bmatrix}
\]

Theorem (Uniqueness of the reduced echelon form)

*Each matrix is row-equivalent to one and only one reduced echelon matrix.*
Definitions

**Definition**

A **pivot position** is the position of a leading entry in an echelon form of the matrix.

**Definition**

A **pivot** is the number in a pivot position.

**Definition**

A **pivot column** is a column that contains a pivot position.

**Note**

There is no more than one pivot in any row. There is no more than one pivot in any column.
Row reduce to **echelon form** and identify the pivots:

\[
\begin{bmatrix}
0 & -3 & -6 & 4 \\
-1 & -2 & -1 & 3 \\
1 & 4 & 5 & -9 \\
\end{bmatrix}
\]

Then find the **reduced echelon form**.
Examples (cont.)
Examples (cont.)
Matlab exercise

Find the **reduced echelon form** in Matlab for:

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}
\]

Command: `rref`
Create a Matlab script that prints on screen the reduced echelon form of matrix A:

\[
A = \begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9 \\
\end{bmatrix}
\]
Basic and free variables

Definition

Any variable that corresponds to a pivot column in the augmented matrix of a system is a **basic variable**. All nonbasic variables are called **free variable**.

Example:

\[
\begin{bmatrix}
1 & 6 & 0 & 3 & 0 & 0 \\
0 & 0 & 1 & -8 & 0 & 5 \\
0 & 0 & 0 & 0 & 1 & 7 \\
\end{bmatrix}
\]

\[
\begin{align*}
x_1 & +6x_2 & +3x_4 & = 0 \\
x_3 & -8x_4 & & = 5 \\
x_5 & & & = 7
\end{align*}
\]

pivot columns:

basic variables:

free variables:
Solutions of linear systems

Using row reduction to solve linear systems

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain the **reduced** echelon form of the augmented matrix.
3. Write the system of equations corresponding to the matrix obtained in step 3.
4. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.
Example

\[
\begin{bmatrix}
  1 & 6 & 0 & 3 & 0 & 0 \\
  0 & 0 & 1 & -8 & 0 & 5 \\
  0 & 0 & 0 & 0 & 0 & 17
\end{bmatrix}
\]

\[
x_1 + 6x_2 + 3x_4 = 0 \\
x_3 - 8x_4 = 5 \\
x_5 = 7
\]

\[
\begin{cases}
x_1 = -6x_2 - 3x_4 \\
x_2 \text{ is free} \\
x_3 = 5 + 8x_4 \\
x_4 \text{ is free} \\
x_5 = 7
\end{cases}
\]

General Solution

The general solution of the system provides a parametric description of the solution set, with the free variables as parameters.

The above system has infinitely many solutions. Why?
Existence and uniqueness

\[
\begin{align*}
  3x_2 &- 6x_3 + 6x_4 + 4x_5 = -5 \\
  3x_1 &- 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\
  3x_1 &- 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 
\end{align*}
\]

An echelon form for the augmented system matrix is:

\[
\begin{bmatrix}
  3 & -9 & 12 & -9 & 6 & 15 \\
  0 & 2 & -4 & 4 & 2 & -6 \\
  0 & 0 & 0 & 0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

(\(x_5 = 4\))

No row has the form

\[
\begin{bmatrix}
  0 & \ldots & 0 & b \\
\end{bmatrix}
\]

where \(b \neq 0\). So, the system is consistent.

Free variables:

\[\text{Consistent system with free variables} \quad \Rightarrow \quad \text{infinitely many solutions}\]
Existence and uniqueness

\[
\begin{align*}
3x_1 + 4x_2 &= -3 \\
2x_1 + 5x_2 &= 5 \\
-2x_1 - 3x_2 &= 1
\end{align*}
\rightarrow \left[
\begin{array}{ccc}
3 & 4 & -3 \\
2 & 5 & 5 \\
-2 & -3 & 1
\end{array}
\right]
\sim
\left[
\begin{array}{ccc}
3 & 4 & -3 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{array}
\right]
\rightarrow
\begin{align*}
3x_1 + 4x_2 &= -3 \\
x_2 &= 3
\end{align*}
\]

Consistent system, no free variables \(\Rightarrow\) unique solution
Theorem (Existence and Uniqueness)

1. A linear system is consistent if and only if an echelon form of the augmented matrix has no row of the form

\[
\begin{bmatrix}
0 & \ldots & 0 & b
\end{bmatrix}
\]

where \(b \neq 0\).

2. If a linear system is consistent, then the set of solutions contains either
   - a unique solution (when there are no free variables)
   - or
   - infinitely many solutions (when there is at least one free variable).
Updated procedure

Using row reduction to solve linear systems

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain the reduced echelon form of the augmented matrix. If the system is NOT consistent, stop. Otherwise, go to the next step.

3. Write the system of equations corresponding to the matrix obtained in step 3.

4. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.
Questions

- What is the largest possible number of pivots a $4 \times 6$ matrix can have? Why?

- What is the largest possible number of pivots a $6 \times 4$ matrix can have? Why?
Questions

- How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

- Suppose the coefficient matrix corresponding to a linear system is $4 \times 6$ and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?
Matlab exercise

Create a Matlab script that prints on screen matrix:

$$\begin{bmatrix}
1 & -2 \\
0 & 7^k
\end{bmatrix}$$

for $k = -1, 0, 1, 2, 3$. 