MATH 2331 - 19859

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Vector equations

Section 1.3
Vector

**Vectors in \( \mathbb{R}^n \)**

Vectors with \( n \) entries: 
\[
\mathbf{u} = \begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_n
\end{bmatrix}, \text{ a matrix with one column.}
\]

**Geometric description of \( \mathbb{R}^2 \)**

Vector \( \mathbf{x} = \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} \) is depicted as the arrow connecting the origin of the axes \( \mathbf{0} = \begin{bmatrix}
    0 \\
    0
\end{bmatrix} \) to the point \( (x_1, x_2) \) in the plane.

\( \mathbb{R}^2 \) is the set of all points in the plane.
Operations with vectors

**Sum**

Given vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \), vector \( \mathbf{u} + \mathbf{v} \in \mathbb{R}^n \) is:

\[
\mathbf{u} + \mathbf{v} = \begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_n
\end{bmatrix} + \begin{bmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_n
\end{bmatrix} = \begin{bmatrix}
    u_1 + v_1 \\
    u_2 + v_2 \\
    \vdots \\
    u_n + v_n
\end{bmatrix}
\]

**Multiplication by scalar**

Given vectors \( \mathbf{u} \in \mathbb{R}^n \), and scalar \( c \in \mathbb{R} \) is:

\[
c\mathbf{u} = c \begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_n
\end{bmatrix} = \begin{bmatrix}
    cu_1 \\
    cu_2 \\
    \vdots \\
    cu_n
\end{bmatrix}
\]
Parallelogram rule

If $u$ and $v$ in $\mathbb{R}^2$ are represented as arrows to points in the plane, then $u + v$ corresponds to the diagonal of the parallelogram with $u$ and $v$ as two sides.

Example: Let $u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $u + v$ is:
Linear combinations of vectors

Given vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \) in \( \mathbb{R}^n \) and given scalars \( c_1, c_2, \ldots, c_p \) in \( \mathbb{R} \), the vector \( \mathbf{y} \) defined by

\[
\mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p
\]

is called a **linear combination** of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \) using weights \( c_1, c_2, \ldots, c_p \).

Example:
Example

Let \( a_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \ a_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}, \ a_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}, \) and \( b = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}. \)

Determine if \( b \) is a linear combination of \( a_1, a_2, \) and \( a_3. \)
Review of the example

\[ a_1, a_2, a_3 \text{ and } b \text{ are columns of the augmented matrix} \]

\[
\begin{bmatrix}
1 & 4 & 3 & -1 \\
0 & 2 & 6 & 8 \\
3 & 14 & 10 & -5 \\
\end{bmatrix}
\]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

\[ a_1 \quad a_2 \quad a_3 \quad b \]

Solution to

\[ x_1 a_1 + x_2 a_2 + x_3 a_3 = b \]

is found by solving the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & a_3 & b \\
\end{bmatrix}.
\]
A vector equation

\[ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_n & b
\end{bmatrix}.
\]

In particular, \( b \) is a linear combination of \( a_1, a_2, \ldots, a_n \) if and only if there is a solution to the linear system corresponding to the augmented matrix.
Span of a set of vectors

\[ \text{Span}\{u\} \] is the set of all vectors of the form \( cu \).

Let \( u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). We have seen that \( u, 2u, \) and \( -u \) lie on the same line. In general, all vectors of the form \( cu \) lie on the same line.

\( \rightarrow \text{Span}\{u\} \) is a line through the origin. We say \( u \) spans \( \mathbb{R} \).
Span of a set of vectors

**Span**{\( u, v \)} is the set of all vectors of the form \( x_1 u + x_2 v \).

Let \( u = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and \( v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). We have seen that \( u, v, \) and \( u + v \) lie on the same plane. In general, all vectors of the form \( x_1 u + x_2 v \) lie on the same plane.

\[ \rightarrow \text{Span}\{u, v\} = \text{a plane through the origin if } v \text{ is NOT a multiple of } u. \] In this case we say \( u, v \) span \( \mathbb{R}^2 \).

What is **Span**{\( u, v \)} if \( v \) is a multiple of \( u \)?
Span of a Set of Vectors: Definition

Span of a Set of Vectors

Suppose $v_1, v_2, \ldots, v_p$ are in $\mathbb{R}^n$; then

$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \text{set of all linear combinations of } v_1, v_2, \ldots, v_p.$$

This means that $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the collection of all vectors that can be written as

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p$$

where $x_1, x_2, \ldots, x_p$ are scalars.
Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $b$ in the span of the columns of $A$?
The Matrix Equation $Ax = b$

Section 1.4
Key concept: linear combinations can be viewed as a matrix-vector multiplication.

Matrix-Vector Multiplication

If $A$ is an $m \times n$ matrix, with columns $a_1, a_2, \ldots, a_n$, and if $x$ is in $\mathbb{R}^n$, then the **product of $A$ and $x$**, denoted by $Ax$, is the linear combination of the columns of $A$ using the corresponding entries in $x$ as weights:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$
Example

Given:

\[ A = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} 7 \\ -6 \end{bmatrix} \]

find \( Ax \). Verify your answer with Matlab.
Matlab exercise

Given:

\[
B = \begin{bmatrix}
1 & -4 & 2 \\
-3 & 1 & 5
\end{bmatrix}, \quad x = \begin{bmatrix}
-5 \\
2
\end{bmatrix},
\]

find \( Bx \) with Matlab.
Example

Write down the system of equations corresponding to the augmented matrix:

\[
\begin{bmatrix}
2 & 3 & 4 & 9 \\
-3 & 1 & 0 & -2
\end{bmatrix}
\]

Then express the system of equations in vector form and finally in the form \( Ax = b \).
Matrix Equation

Three Equivalent Ways

1. **System of linear equations.**
2. **Vector equation** \( x_1a_1 + x_2a_2 + \cdots + x_na_n = b. \)
3. **Matrix equation** \( Ax = b. \)
Chapter 1

Matrix Equation: Theorem

**Theorem**

If $A$ is a $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if $b$ is in $\mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}.$$

**Note**

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$. 