MATH 2331 - 17571

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Lecture: TuTh 1:00PM-2:30PM in SEC 203
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Questions

- Can a linear system of 4 equations and only 3 unknowns have a unique solution?

- Suppose the coefficient matrix corresponding to a linear system is $4 \times 6$ and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?
Vector equations

Section 1.3
Vectors in $\mathbb{R}^n$

Vectors with $n$ entries: $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, a matrix with one column.

Geometric description of $\mathbb{R}^2$

Vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is depicted as the arrow connecting the origin of the axes $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the point $(x_1, x_2)$ in the plane.

$\mathbb{R}^2$ is the set of all points in the plane.
Operations with vectors

### Sum
Given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, vector $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$ is:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

### Multiplication by scalar
Given vectors $\mathbf{u} \in \mathbb{R}^n$, and scalar $c \in \mathbb{R}$ is:

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$
Parallelogram rule

If \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^2 \) are represented as arrows to points in the plane, then \( \mathbf{u} + \mathbf{v} \) corresponds to the diagonal of the parallelogram with \( \mathbf{u} \) and \( \mathbf{v} \) as two sides.

Example: Let \( \mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). \( \mathbf{u} + \mathbf{v} \) is:
Linear combinations of vectors

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$ in $\mathbb{R}$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

Example:
Linear combinations and vector equation

Vector Equation

A vector equation

\[ x_1a_1 + x_2a_2 + \cdots + x_na_n = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
  a_1 & a_2 & \cdots & a_n & b
\end{bmatrix}.
\]

In particular, \( b \) is a linear combination of \( a_1, a_2, \ldots, a_n \) if and only if there is a solution to the linear system corresponding to the augmented matrix.
Span of a set of vectors

\[ \text{Span}\{u\} \text{ is the set of all vectors of the form } cu. \]

Let \( u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). We have seen that \( u, 2u, \) and \( -u \) lie on the same line. In general, all vectors of the form \( cu \) lie on the same line.

\[ \rightarrow \text{Span}\{u\} \text{ is a line through the origin. We say } u \text{ spans } \mathbb{R}. \]
Span of a set of vectors

Span\{u, v\} is the set of all vectors of the form \(x_1u + x_2v\).

Let \(u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\) and \(v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\). We have seen that \(u\), \(v\), and \(u + v\) lie on the same plane. In general, all vectors of the form \(x_1u + x_2v\) lie on the same plane.

→ Span\{u, v\} = a plane through the origin if \(v\) is NOT a multiple of \(u\). In this case we say \(u, v\) span \(\mathbb{R}^2\).

What is Span\{u, v\} if \(v\) is a multiple of \(u\)?
Span of a Set of Vectors: Definition

**Span of a Set of Vectors**

Suppose \( v_1, v_2, \ldots, v_p \) are in \( \mathbb{R}^n \); then

\[
\text{Span}\{v_1, v_2, \ldots, v_p\} = \text{set of all linear combinations of } v_1, v_2, \ldots, v_p.
\]

This means that \( \text{Span}\{v_1, v_2, \ldots, v_p\} \) is the collection of all vectors that can be written as

\[
x_1 v_1 + x_2 v_2 + \cdots + x_p v_p
\]

where \( x_1, x_2, \ldots, x_p \) are scalars.
Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $b$ in the span of the columns of $A$?
The Matrix Equation $Ax = b$

Section 1.4
Key concept: linear combinations can be viewed as a matrix-vector multiplication.

Matrix-Vector Multiplication

If $A$ is an $m \times n$ matrix, with columns $a_1, a_2, \ldots, a_n$, and if $x$ is in $\mathbb{R}^n$, then the product of $A$ and $x$, denoted by $Ax$, is the linear combination of the columns of $A$ using the corresponding entries in $x$ as weights:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$
Example

Given:

\[ A = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} 7 \\ -6 \end{bmatrix} \]

find \( Ax \). Verify your answer with Matlab.
Matlab exercise

Given:

\[ B = \begin{bmatrix} 1 & -4 & 2 \\ -3 & 1 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \]

find \( Bx \) with Matlab.
Example

Write down the system of equations corresponding to the augmented matrix:

\[
\begin{bmatrix}
2 & 3 & 4 & 9 \\
-3 & 1 & 0 & -2 \\
\end{bmatrix}
\]

Then express the system of equations in vector form and finally in the form \( Ax = b \).
Example (cont.)
Matrix Equation

Three Equivalent Ways

2. Vector equation $x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$.
3. Matrix equation $Ax = b$. 
Matrix Equation: Theorem

**Theorem**

If $A$ is a $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if $b$ is in $\mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}.$$

**Note**

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$. 