MATH 2331 - 17571

Vladimir Yushutin
yushutin@math.uh.edu
Office: PGH 606
Lecture: TuTh 1:00PM-2:30PM in SEC 203
Office hours: TuTh 2.30PM-4PM and BY APPOINTMENT
Linear combinations of vectors

Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$ in $\mathbb{R}$, the vector $y$ defined by

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

is called a linear combination of $v_1, v_2, \ldots, v_p$ using weights $c_1, c_2, \ldots, c_p$.

Span of a Set of Vectors

Suppose $v_1, v_2, \ldots, v_p$ are in $\mathbb{R}^n$; then

$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \text{set of all linear combinations of } v_1, v_2, \ldots, v_p.$$
Matrix-Vector Multiplication

If $A$ is an $m \times n$ matrix, with columns $a_1, a_2, \ldots, a_n$, and if $x$ is in $\mathbb{R}^n$, then the **product of $A$ and $x$**, denoted by $Ax$, is the linear combination of the columns of $A$ using the corresponding entries in $x$ as weights:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$
Matrix Equation

Three Equivalent Ways

2. Vector equation \( x_1a_1 + x_2a_2 + \cdots + x_na_n = b \).
3. Matrix equation \( Ax = b \).
Matrix Equation: Theorem

**Theorem**

If $A$ is a $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if $b$ is in $\mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}.$$

**Note**

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$. 
Example

Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $Ax = b$ consistent for all $b$?
Example

In the previous example, we considered matrix

\[
A = \begin{bmatrix}
1 & 4 & 5 \\
-3 & -11 & -14 \\
2 & 8 & 10 \\
\end{bmatrix}
\]

Notice that the columns of \( A \) are not independent (i.e., one of them is a linear combination of the others): the third column of \( A \) is the first column plus the second column → the columns of \( A \) span \( \mathbb{R}^2 \), NOT \( \mathbb{R}^3 \).

This is the reason why \( Ax = b \) is not consistent for all \( b \).
Matrix Equation: Span $\mathbb{R}^m$

**Definition**

We say that the columns of $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_p \end{bmatrix}$ span $\mathbb{R}^m$ and write

$$\text{Span } \{a_1, \ldots, a_p\} = \mathbb{R}^m$$

if every vector $b$ in $\mathbb{R}^m$ is a linear combination of $a_1, \ldots, a_p$.

**Theorem**

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.
2. Each $b$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.
3. The columns of $A$ span $\mathbb{R}^m$.
4. $A$ has a pivot position in every row.
Example

Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \) and \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \). Is the equation \( Ax = b \) consistent for all possible \( b \)?
Example

Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbb{R}^3$?
Theorem

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
2. $A(c\mathbf{u}) = cA\mathbf{u}$. 
Solutions sets of linear systems

Section 1.5
Definition

Let $A$ be a $m \times n$ matrix. System $Ax = 0$, where $0$ is the zero vector in $\mathbb{R}^m$, is called **homogeneous system**.

Example:

\[
\begin{align*}
    x_1 &+ 10x_2 &= 0 \\
    2x_1 &+ 20x_2 &= 0
\end{align*}
\]

The homogeneous system $Ax = 0$ *always* has the **trivial solution**: $x = 0$. 
Homogeneous System: Nontrivial Solutions

Definition
Nonzero vector solutions are called **nontrivial solutions**.

Does the system in the previous slide have **nontrivial** solutions?

A homogeneous equation $Ax = 0$ has nontrivial solutions if and only if the system of equations has infinitely many solutions, i.e., if the system has at least one free variable.
Example

Determine if the following homogeneous system has nontrivial solutions and then describe the solution set:

\[ 2x_1 + 4x_2 - 6x_3 = 0 \]
\[ 4x_1 + 8x_2 - 10x_3 = 0 \]
Example (cont.)

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= \begin{bmatrix}
  -2x_2 \\
  x_2 \\
  0
\end{bmatrix}
= x_2 \begin{bmatrix}
  -2 \\
  1 \\
  0
\end{bmatrix}
= x_2 v
\]

Graphical representation:

solution set = \text{span}\{v\} = \text{line through 0 in } \mathbb{R}^3
Nonhomogeneous System

Definition
Let $A$ be a $m \times n$ matrix. System $Ax = b$, where $b \neq 0$, is called nonhomogeneous system.

Example: Describe the solution set of

\[
\begin{align*}
2x_1 + 4x_2 - 6x_3 &= 0 \\
4x_1 + 8x_2 - 10x_3 &= 4
\end{align*}
\]
Example (cont.)

\[
x = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = p + x_2v
\]

Graphical representation: line parallel to \(v\) passing through the tip of vector \(p\).
Recap of the previous examples

- Solution of $Ax = 0$:
  \[ x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2v \]
  \[ x = x_2v = \text{parametric equation of line passing through 0 and } v \]

- Solution of $Ax = b$:
  \[ x = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = p + x_2v \]
  \[ x = p + x_2v = \text{parametric equation of line passing through the tip of } p \text{ and parallel to } v \]

The solution sets of $Ax = 0$ and $Ax = b$ are parallel.