Linear Independence

Section 1.7
Linear Independence: Definition

**Linear Independence**

A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution.

**Linear Dependence**

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists weights \( c_1, \ldots, c_p \), NOT ALL 0, such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]
Example

Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \).

a. Determine if \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is linearly independent.
b. If possible, find a linear dependence relation among \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).
Example (cont.)
Linear Independence of Matrix Columns

Notice that

\[
-33 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

can be written as the matrix equation:

\[
\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Each linear dependence relation among the columns of \( A \) corresponds to a nontrivial solution to \( A\mathbf{x} = 0 \).

The columns of matrix \( A \) are linearly independent if and only if the equation \( A\mathbf{x} = 0 \) has only the trivial solution.
Special Cases: 1. A Set of One Vector

Sometimes we can determine linear independence of a set with minimal effort.

1. A Set of One Vector

**Example:** Consider the set \( \{v_1\} \) with

\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}.
\]

Is it is linearly independent?

\( \{v_1\} \) is linearly independent when \( v_1 \neq 0 \).
2. A Set of Two Vectors

**Example:** Let

\[
\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.
\]

a. Determine if \(\{\mathbf{u}_1, \mathbf{u}_2\}\) is a linearly dependent set or a linearly independent set.
b. Determine if \(\{\mathbf{v}_1, \mathbf{v}_2\}\) is a linearly dependent set or a linearly independent set.
A Set of Two Vectors (cont.)
A set of two vectors is linearly dependent if one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.
Theorem

A set of vectors \( S = \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) containing the zero vector is linearly dependent.

Proof:
Special Cases: 4. A Set Containing Too Many Vectors

**Theorem**

*If a set contains more vectors than entries in each vector, then the set is linearly dependent, i.e. any set \{v_1, v_2, \ldots, v_p\} in \mathbb{R}^n is linearly dependent if p > n.*

**Outline of Proof:**

\[ A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \text{ is } n \times p. \]
Special Cases: Examples

With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. \[
\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}
\]

b. Columns of
\[
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}
\]
Examples (cont.)

c. \{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}

d. \{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \}
Example

Consider a set of vectors \( \{v_1, v_2, v_3, v_4\} \) in \( \mathbb{R}^3 \). Is the set linearly dependent? Explain why.
**Linear Independence:**

A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution.

**Linear Dependence:**

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists weights \( c_1, \ldots, c_p \), NOT ALL 0, such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]
Linear Independence of Matrix Columns

Each linear dependence relation among the columns of \( A \) corresponds to a nontrivial solution to \( Ax = 0 \).

The columns of matrix \( A \) are linearly independent if and only if the equation \( Ax = 0 \) has only the trivial solution.

Special cases:

1. A Set of One Vector: \( \{ v_1 \} \) is linearly independent when \( v_1 \neq 0 \).
2. A Set of Two Vectors: linearly independent if and only if neither of the vectors is a multiple of the other.
3. A set of vectors \( S = \{ v_1, v_2, \ldots, v_p \} \) in \( \mathbb{R}^n \) containing the zero vector is linearly dependent.
4. If a set contains more vectors than entries in each vector, then the set is linearly dependent.
Characterization of Linearly Dependent Sets

Theorem

A set \( S = \{v_1, v_2, \ldots, v_p\} \) of two or more vectors is linearly dependent if and only if at least one vector in \( S \) is a linear combination of the others. In fact, if \( S \) is linearly dependent, and \( v_1 \neq \mathbf{0} \), then some vector \( v_j \ (j \geq 2) \) is a linear combination of the preceding vectors \( v_1, \ldots, v_{j-1} \).