MATH 2331 - 20947

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Lecture : MoWeFr 10:00AM-11:00AM in CBB 104
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Matrix Equation

Matrix-Vector Multiplication

\[ Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \]

Note

The equation \( Ax = b \) has a solution if and only if \( b \) is a linear combination of the columns of \( A \).

Three Equivalent Ways

2. Vector equation \( x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b \).
3. Matrix equation \( Ax = b \).
Matrix Equation: Theorem

Theorem

If $A$ is a $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if $b$ is in $\mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}.$$
Example

Let \( A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \).

Is the equation \( A\mathbf{x} = \mathbf{b} \) consistent for all \( \mathbf{b} \)?
In the previous example, we considered matrix

\[
A = \begin{bmatrix}
1 & 4 & 5 \\
-3 & -11 & -14 \\
2 & 8 & 10 \\
\end{bmatrix}
\]

Notice that the columns of \( A \) are not independent (i.e., one of them is a linear combination of the others): the third column of \( A \) is the first column plus the second column

\[ \rightarrow \text{the columns of } A \text{ span } \mathbb{R}^2, \text{ NOT } \mathbb{R}^3. \]

This is the reason why \( Ax = b \) is not consistent for all \( b \).
Matrix Equation: Span $\mathbb{R}^m$

**Definition**

We say that the columns of $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_p \end{bmatrix}$ span $\mathbb{R}^m$ and write

$$\text{Span} \{a_1, \ldots, a_p\} = \mathbb{R}^m$$

if every vector $b$ in $\mathbb{R}^m$ is a linear combination of $a_1, \ldots, a_p$.

**Theorem**

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.
2. Each $b$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.
3. The columns of $A$ span $\mathbb{R}^m$.
4. $A$ has a pivot position in every row.
Example

Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \) and \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \). Is the equation \( Ax = b \) consistent for all possible \( b \)?
Example

Do the columns of \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \) span \( \mathbb{R}^3 \)?
Theorem

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
2. $A(c\mathbf{u}) = cA\mathbf{u}$. 