MATH 2331 - 19859

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**Linear Independence: Definition**

**Linear Independence**
A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution.

**Linear Dependence**
The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists weights \( c_1, \ldots, c_p, \) NOT ALL 0, such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]
Linear Independence of Matrix Columns

Each linear dependence relation among the columns of $A$ corresponds to a nontrivial solution to $Ax = 0$.

The columns of matrix $A$ are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

Special cases:

1. A Set of One Vector: $\{v_1\}$ is linearly independent when $v_1 \neq 0$.
2. A Set of Two Vectors: linearly independent if and only if neither of the vectors is a multiple of the other.
3. A set of vectors $S = \{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$ containing the zero vector is linearly dependent.
4. If a set contains more vectors than entries in each vector, then the set is linearly dependent.
Theorem

A set \( S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \) of two or more vectors is linearly dependent if and only if at least one vector in \( S \) is a linear combination of the others. In fact, if \( S \) is linearly dependent, and \( \mathbf{v}_1 \neq \mathbf{0} \), then some vector \( \mathbf{v}_j \) \((j \geq 2)\) is a linear combination of the preceding vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_{j-1} \).
Theorem

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.
2. Each $b$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.
3. Matrix $A$ has a pivot position in every row.
4. The columns of $A$ span $\mathbb{R}^m$. 
Introduction to Linear Transformations

Section 1.8
Matrix Transformations

Another Way to View $Ax = b$

Matrix $A$ is an object acting on $x$ to produce a new vector $b$.

Matrix Transformations

Suppose $A$ is $m \times n$. Solving $Ax = b$ amounts to finding all $x$ in $\mathbb{R}^n$ which are transformed into vector $b$ in $\mathbb{R}^m$ through multiplication by $A$. 

\[ \text{multiply by } A \quad \text{transformation "machine"} \]
A transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $x$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. 

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$
Consider transformation:

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \]
\[ T : x \rightarrow T(x) \]

\( \mathbb{R}^n \) is the **domain** of \( T \).

\( \mathbb{R}^m \) is the **codomain** of \( T \).

\( T(x) \) in \( \mathbb{R}^m \) is the **image** of \( x \) under the transformation \( T \).

Set of all images \( T(x) \) is the **range** of \( T \).
Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $T(x) = Ax$.

Given $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find $T(x)$. 
Example (cont.)
Example

Given:

\[ A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -10 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \]

define a transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) by \( T(x) = Ax \).

a. Find an \( x \) in \( \mathbb{R}^3 \) whose image under \( T \) is \( b \).
Example (cont.)

Given:

\[ A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -10 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \]

define a transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by \( T(x) = Ax \).

b. Is there more than one \( x \) under \( T \) whose image is \( b \)? (uniqueness problem)
Example (cont.)

Given:

\[
A = \begin{bmatrix}
1 & -2 & 3 \\
-5 & 10 & -15
\end{bmatrix},
\quad
b = \begin{bmatrix}
2 \\
-10
\end{bmatrix},
\quad
\mathbf{c} = \begin{bmatrix}
3 \\
0
\end{bmatrix},
\]

define a transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by \( T(\mathbf{x}) = A\mathbf{x} \).

c. Determine if \( \mathbf{c} \) is in the range of the transformation \( T \).

\textit{(existence problem)}
Matrix transformations have many applications - including computer graphics.
Linear Transformations

If $A$ is $m \times n$, then the transformation $T(x) = Ax$ has the following properties:

\[ T(u + v) = A(u + v) = \ldots + \ldots \]

\[ = \ldots + \ldots \]

and

\[ T(cu) = A(cu) = \ldots Au = \ldots T(u) \]

for all $u, v$ in $\mathbb{R}^n$ and all scalars $c$.

Linear Transformation

A transformation $T$ is linear if:

1. $T(u + v) = T(u) + T(v)$ for all $u, v$ in the domain of $T$.
2. $T(cu) = cT(u)$ for all $u$ in the domain of $T$ and all scalars $c$. 
Every matrix transformation is a linear transformation.

**Theorem**

If $T$ is a linear transformation, then

$$T(0) = 0 \quad \text{and} \quad T(cu + dv) = cT(u) + dT(v).$$

**Proof:**

$$T(0) = T(0u) =$$

$$T(cu + dv) = T(\quad ) + T(\quad ) =$$
Example

Let \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ y_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \) and \( y_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \).

Suppose \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a linear transformation which maps \( e_1 \) into \( y_1 \) and \( e_2 \) into \( y_2 \). Find the images of \( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).
Example (cont.)

\[ T(3e_1 + 2e_2) = 3T(e_1) + 2T(e_2) \]
Example (cont.)
Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ such

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Show that $T$ is not a linear transformation.

A way to solve the problem is to provide a counterexample. For instance, you can show that $T(0) \neq 0$