Linear Independence

Section 1.7
Linear Independence: Definition

**Linear Independence**

A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution.

**Linear Dependence**

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exist weights \( c_1, \ldots, c_p \), NOT ALL 0, such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]
Special Cases: 1. A Set of One Vector

Sometimes we can determine linear independence of a set with minimal effort.

1. A Set of One Vector

**Example:** Consider the set \( \{v_1\} \) with

\[
\begin{bmatrix}
1 \\
0 \\
-5
\end{bmatrix}
\]

Is it a linearly independent set?

\( \{v_1\} \) is linearly independent when \( v_1 \neq 0 \).
2. A Set of Two Vectors

Example: Let

\[ \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

a. Determine if \( \{ \mathbf{u}_1, \mathbf{u}_2 \} \) is a linearly dependent set or a linearly independent set.

b. Determine if \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is a linearly dependent set or a linearly independent set.
A set of two vectors is linearly dependent if one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.
A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one vector in $S$ is a linear combination of the others.
Theorem

A set of vectors $S = \{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$ containing the zero vector is linearly dependent.

Proof:
Special Cases: 4. A Set Containing Too Many Vectors

**Theorem**

If a set contains more vectors than entries in each vector, then the set is linearly dependent, i.e. any set \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} \) in \( \mathbb{R}^n \) is linearly dependent if \( p > n \).

**Outline of Proof:**

\[
A = \begin{bmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p
\end{bmatrix}
\text{ is } n \times p.
\]
Special Cases: Examples

With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. \( \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\} \)

b. Columns of
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 0 \\
9 & 8 & 7 & 6 & 5 \\
4 & 3 & 2 & 1 & 8 \\
\end{bmatrix}
\]
Examples (cont.)

c. \[ \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \]

d. \[ \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\} \]
Consider a set of vectors \( \{v_1, v_2, v_3, v_4\} \) in \( \mathbb{R}^3 \). Is the set linearly dependent? Explain why.