Exam 1 will be on Thursday 09/27 IN CLASS (75 min.).
Exam 1 will be on Sec. 1.1-1.9.
Review for the exam on Tuesday 02/25.
A transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $x$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ is the **domain** of $T$.

$\mathbb{R}^m$ is the **codomain** of $T$.

$T(x)$ in $\mathbb{R}^m$ is the **image** of $x$ under the transformation $T$.

Set of all images $T(x)$ is the **range** of $T$. 
Example

Given:

\[ A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -10 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \]

define a matrix transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) by \( T(x) = Ax \).

a. Find an \( x \) in \( \mathbb{R}^3 \) whose image under \( T \) is \( b \).
Example (cont.)

Given:

\[ A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -10 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \]

define a transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) by \( T(x) = Ax \).

b. Is there more than one \( x \) under \( T \) whose image is \( b \)? *(uniqueness problem)*
Example (cont.)

Given:

\[ A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -10 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \]

define a transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by \( T(x) = Ax \).

c. Determine if \( c \) is in the range of the transformation \( T \).
(\textit{existence problem})
Linear Transformations

A transformation $T$ is linear if:

1. $T(u + v) = T(u) + T(v)$ for all $u, v$ in the domain of $T$.
2. $T(cu) = cT(u)$ for all $u$ in the domain of $T$ and all scalars $c$.

Examples: rotation, scaling, shearing but not translation.
Matrix Transformations

If $A$ is $m \times n$, then the transformation $T(x) = Ax$ has the following properties:

$$T(u + v) = A(u + v) = _____ + _____$$

$$= _____ + _____$$

and

$$T(cu) = A(cu) = _____Au = _____ T(u)$$

for all $u, v$ in $\mathbb{R}^n$ and all scalars $c$.

Every matrix transformation is a **linear** transformation.
Theorem

If $T$ is a linear transformation, then

$$T(0) = 0 \quad \text{and} \quad T(cu + dv) = cT(u) + dT(v).$$

Proof:

$$T(0) = T(0u) =$$

$$T(cu + dv) = T(\phantom{v}) + T(\phantom{v}) =$$
Example

Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $y_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $y_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation which maps $e_1$ into $y_1$ and $e_2$ into $y_2$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. 
Example (cont.)

\[ T(3e_1 + 2e_2) = 3T(e_1) + 2T(e_2) \]
Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ such

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Show that $T$ is not a linear transformation.

A way to solve the problem is to provide a counterexample. For instance, you can show that $T(0) \neq 0$
Identity Matrix

$I_n$ is an $n \times n$ matrix with ones on the main (left to right diagonal) and zeros elsewhere. The $i$-th column of $I_n$ is labeled $e_i$.

Identity Matrix

In general, for $x$ in $\mathbb{R}^n$: $I_n x = x$

Matlab command: `eye(n)`
Matrix of Linear Transformation: Theorem

**Theorem**

Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Then there exists a unique matrix \( A \) such that

\[
T(x) = Ax
\]

for all \( x \) in \( \mathbb{R}^n \).

Let \( e_j \) is the \( j \)-th column of the identity matrix in \( \mathbb{R}^n \). Matrix \( A \) is the \( m \times n \) matrix whose \( j \)-th column is the vector \( T(e_j) \):

\[
A = [T(e_1) \quad T(e_2) \quad \cdots \quad T(e_n)]
\]

**Definition**

Such matrix \( A \) is called standard matrix of the linear transformation \( T \).
Consider transformation:

\[ T(x) = T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix} \]

Find the standard matrix of \( T \).
Example (cont.)
Example

Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates a point about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).

$T(e_1) = \begin{bmatrix} \end{bmatrix} \quad T(e_2) = \begin{bmatrix} \end{bmatrix}$