MATH 2331 - 17571

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Lecture : TuTh 1:00PM-2:30PM in SEC 203
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The Inverse of a Matrix

Section 2.2
The Inverse of a Matrix: Definition

The inverse of a real number $a$ is denoted by $a^{-1}$. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1.$$ 

The Inverse of a Matrix

An $n \times n$ matrix $A$ is said to be invertible if there is an $n \times n$ matrix $C$ satisfying

$$CA = AC = I_n$$

where $I_n$ is the $n \times n$ identity matrix. We call $C$ the inverse of $A$ and denote it $A^{-1}$. 
The Inverse of a 2-by-2 Matrix

Theorem

Let

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]

If \( ad - bc \neq 0 \), then \( A \) is invertible and

\[ A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \]

If \( ad - bc = 0 \), then \( A \) is not invertible.
The Inverse of a Matrix: Definition

Theorem

*If A is an invertible $n \times n$ matrix, then for each $b$ in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.***
The Inverse of a Matrix

**WARNING**

*Not all* $n \times n$ matrices are invertible. A matrix which is *not* invertible is called **singular**. An invertible matrix is called **nonsingular** matrix.

**Fact**

If $A$ is invertible, then the inverse is unique.

**Proof:** Assume $B$ and $C$ are both inverses of $A$. Then

$$B = BI =$$
The Inverse of a Matrix: Theorem

Theorem

Suppose $A$ and $B$ are invertible. Then the following results hold:

a. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$

b. $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof:
Example

Given

\[ A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}, \]

find \( A^{-1} \).
Example

Find the inverse of

\[ B = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix}. \]
Matlab command to find $A^{-1}$: \texttt{inv(A)}

In Matlab, if $A$ is an invertible $n \times n$ matrix the solution of $Ax = b$ is not ordinarily obtained by computing the inverse of $A$, that is:

$$x = \texttt{inv(A)} * b$$

Instead, we use:

$$x = A \backslash b$$

because \texttt{inv} can be slower and less accurate than \texttt{\}. 
The Inverse of Elementary Matrix

**Question**: How do we find the inverse of an invertible $n \times n$ matrix?

To answer this question, we first look at *elementary* matrices.

**Elementary Matrices**

An *elementary matrix* is one that is obtained by performing a single elementary row operation on an identity matrix.

**Example**:

$$ E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$
Multiplication by Elementary Matrices

Given matrix

\[ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \]

Observe the following product and describe how these product can be obtained by elementary row operations on \( A \).

\[ EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix} \]
If an elementary row operation is performed on an $m \times n$ matrix $A$, the resulting matrix can be written as $EA$, where the $m \times m$ matrix $E$ is created by performing the same row operations on $I_m$. 
Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix $E$, determine the elementary row operation needed to transform $E$ back into $I$ and apply this operation to $I$ to find the inverse.

**Example:** Given

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $E^{-1}$. 
The elementary row operations that row reduce $A$ to $I_n$ are the same elementary row operations that transform $I_n$ into $A^{-1}$.

**Theorem**

An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to the identity matrix $I_n$.

In this case, any sequence of elementary row operations that reduces $A$ to $I_n$ will also transform $I_n$ to $A^{-1}$.
Algorithm for Finding $A^{-1}$

1. Place $A$ and $I$ side-by-side to form an augmented matrix $[A \ I]$.
2. Perform row operations on this matrix (which will produce identical operations on $A$ and $I$) to reduce $A$ to the identity matrix.
3. By the Theorem we have just seen:
   
   $$[A \ I] \text{ will row reduce to } [I \ A^{-1}]$$

   or $A$ is not invertible.
Example

If it exists, find the inverse of:

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
-3 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}.
\]

Verify your answer with Matlab.

\[
[A \ I] = \begin{bmatrix}
2 & 0 & 0 & 1 & 0 & 0 \\
-3 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
Example

Suppose $A, B, C,$ and $D$ are invertible $n \times n$ matrices and

$$A = B(D - I_n)C.$$ 

Solve for $D$ in terms of $A, B, C$ and $I_n$.

Be careful!
Order of multiplication is important.
Exercise

Express \((ABC)^{-1}\) in terms of \(A^{-1}\), \(B^{-1}\), and \(C^{-1}\).