MATH 2331 - 17571

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Lecture: TuTh 1:00PM-2:30PM in SEC 203
Office hours: TuTh 2:30PM-4PM and BY APPOINTMENT
Determinants: Cofactor Expansion

Definition

Let $A = [a_{ij}]$ be $n \times n$ matrix, with $n \geq 2$. Then:

$$
\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}
$$

Definition

The $(i, j)$-cofactor of $A$ is an entry of $n \times n$ matrix $C_A$ where

$$(C_A)_{ij} = (-1)^{i+j} \det A_{ij}.$$ 

Example: cofactor expansion across row 1

$$
\begin{vmatrix}
1 & 2 & 0 \\
3 & -1 & 2 \\
2 & 0 & 1
\end{vmatrix}
= 1C_{11} + 2C_{12} + 0C_{13}
$$
Cofactor Expansion: Theorem

Theorem (Cofactor Expansion)

The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column:

- **expansion across row $i$:**
  \[
  \det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}
  \]

- **expansion down column $j$:**
  \[
  \det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}
  \]
Example

Find the determinant of $A$ using cofactor expansion down column 3:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$
Example

Find the determinant of:

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 2 & 1 & 5 \\
0 & 0 & 2 & 1 \\
0 & 0 & 3 & 5
\end{bmatrix}.
\]
Particular Case: Triangular Matrices

### Triangular Matrices

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(upper triangular)

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(lower triangular)

### Theorem

*If A is a triangular matrix, then det A is the product of the main diagonal entries of A.*
Example

\[ \text{det} \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -3 & 0 \\
0 & 0 & 0 & 4 \\
\end{bmatrix} = \]
Properties of Determinants

Section 3.2
Determinant of the Transpose

Theorem

If $A$ is an $n \times n$ matrix with a row (or column) of zeros, then $\det A = 0$.

Theorem

If $A$ is an $n \times n$ matrix, then $\det A^T = \det A$.

Proof:
Properties of Determinants

Theorem (Elementary Row Operations)

Let $A$ be a square matrix.

a. If a multiple of one row of $A$ is added to another row of $A$ to produce a matrix $B$, then $\det A = \det B$.

b. If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.

c. If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \cdot \det A$.

Notice: the theorem still holds if the word row is replaced with column.

Proof:
Example

\[
\begin{vmatrix}
2 & 5 & 7 \\
6 & 11 & 13 \\
-8 & -12 & -12
\end{vmatrix} =
\]
Suppose $A$ has been reduced to

$$U = \begin{bmatrix}
\text{■} & * & * & \cdots & * \\
0 & \text{■} & * & \cdots & * \\
0 & 0 & \text{■} & \cdots & * \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & 0 & \text{■}
\end{bmatrix}$$

by row replacements and row interchanges. Then

$$\det A = (-1)^r \text{ (product of diagonal entries of } U)$$

where $r$ is the number of row interchanges.

**Definition**

Number of pivots columns is called *rank* of matrix $A$ and denoted as $\text{rk}(A)$. 
Invertability and Determinant

(the Conjecture was correct!)

**Theorem**

A square matrix $A$ is invertible if and only if $\det A \neq 0$.

**Proof:**
Example: Partitioned matrix

**Theorem**

Determinant of a \((n + m) \times (n + m)\) matrix \[
\begin{bmatrix}
A & B \\
0 & D
\end{bmatrix}
\] with \(n \times n\) matrix \(A\) and \(m \times m\) matrix \(D\) can be computed as \(\text{det } A \cdot \text{det } D\).

\[
\begin{vmatrix}
1 & 1 & 2 & 3 \\
1 & 1 & 8 & 3 \\
0 & 0 & 2 & -3 \\
0 & 0 & 4 & -9
\end{vmatrix}
= 
\]
Theorem (Multiplicative Property)

For $n \times n$ matrices $A$ and $B$, $\det(AB) = (\det A)(\det B)$.

Example: Find $\det A^3$ if $\det A = 5$. 
Example

For $n \times n$ matrices $A$ and $B$, show that $A$ is singular if $B$ is invertable and $AB$ is not invertable.
Example

\[
\begin{vmatrix}
1 & 2 & 3 & 4 \\
0 & 5 & 0 & 0 \\
2 & 7 & 6 & 10 \\
2 & 9 & 7 & 11
\end{vmatrix} =
\]
Example

\[
\text{det} \begin{bmatrix}
-6 & -12 & -18 & 6 \\
7 & 10 & 13 & 7 \\
5 & 6 & 7 & 9 \\
1 & 4 & 9 & 16 \\
\end{bmatrix} =
\]
Theorem

For a square matrix $Ax = b$, where $A$ is an invertable matrix, the inverse $A^{-1}$ can be computed as

$$A^{-1} = \frac{1}{\det A} C_A^T$$

where $C_A$ is a cofactor matrix of $A$.

Theorem (Cramer’s rule)

For a square linear system $Ax = b$, where $A = \{a_1, \ldots, a_i, \ldots, a_n\}$ is an invertable matrix, the solution $x$ can be computed as

$$x_i = \frac{\det A_i}{\det A}$$

where $A_i = \{a_1, \ldots, b, \ldots, a_n\}$. 
Example

By row reduction and cofactor expansion find:

\[
\begin{vmatrix}
2 & 3 & 0 & 1 \\
4 & 7 & 0 & 3 \\
7 & 9 & -2 & 4 \\
1 & 2 & 0 & 4 \\
\end{vmatrix}
= \]

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Plotting Graphs in Matlab
The command to trace a graph is:

plot(x, y)

where $x$ is a vector containing the values on the horizontal axis and $y$ is a vector containing the values on the vertical axis.

**Example:**
to plot the graph of $\sin(x)$ from 0 to $2\pi$ use commands:

```matlab
x = 0:pi/100:2*pi;
y = sin(x);
plot(x,y)
```
To add plots to an existing figure, use `hold on`.

**Example:**
to add the graph of $\cos(x)$ from 0 to $2\pi$ use commands:

```
hold on
y2 = cos(x);
plot(x, y2, 'r--')
```

By using 'r--' we plot the curve with a dashed red line.

For more details, type:

```
help plot
```
Matlab exercise

Plot functions $f(x) = x^2$ and $g(x) = x^3$ for $x$ in $[-1, 1]$. 
You can also use the logarithmic scale for one or both axes with commands

- `semilogy(x,y)`: logarithmic scale for the vertical axis;
- `semilogx(x,y)`: logarithmic scale for the horizontal axis;
- `loglog(x,y)`: logarithmic scale for both axes.

**Example:**
Plot $y = e^x$ for $x$ in $[0, 2]$ in logarithmic scale for the vertical axis.

```matlab
x = 0:0.001:2;
y = exp(x);
semilogy(x,y)
```

Notice that this is the same as:

```matlab
plot(x, log(y))
```