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Lecture: MoWeFr 10:00AM-11:00AM in CBB 104
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The Invertible Matrix Theorem

Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent (i.e., for a given $A$, they are either all true or all false).

a. $A$ is an invertible matrix.

j. There are $n \times n$ matrices $C$, $D$ such that $CA = I_n$ and $AD = I_n$.

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot positions.

l. $A^T$ is an invertible matrix.
The Invertible Matrix Theorem (cont.)

d. The equation $Ax = 0$ has only the trivial solution.

e. The columns of $A$ form a linearly independent set.

g. The equation $Ax = b$ is consistent for each $b$ in $\mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

f. The linear transformation $x \rightarrow Ax$ is one-to-one.

i. The linear transformation $x \rightarrow Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. 
Use the Invertible Matrix Theorem to determine if $A$ is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 1 & 11 & 1 \\ 2 & 7 & 2 \end{bmatrix}.$$
Example

Is it possible for a $4 \times 4$ matrix $A$ to be invertible when its columns do not span $\mathbb{R}^4$? Why or why not?
Example

If an $n \times n$ matrix $A$ is invertible, then the columns of $A^T$ are linearly independent. Explain why.
Can a square matrix $A$ with two identical columns be invertible? Why or why not?
Example

Can a square matrix $A$ with two identical rows be invertible? Why or why not?
Example

If the columns of a $7 \times 7$ matrix $D$ are linearly independent, what can be said about the solutions of $Dx = b$? Why?
Introduction to Determinants

Section 3.1
Determinant: "invertibility" number

Determinant of a $1 \times 1$ matrix

Given a $1 \times 1$ matrix $A = a$, with $a \in \mathbb{R}$, $\det A = a$.

Determinant of a $2 \times 2$ matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$  

Then, $\det A = ad - bc$.

Determinant of a $n \times n$ matrix? Conjecture:

There is a number associated with any matrix (function $\det()$ on matrices), zero value of which corresponds to a singular matrices only.
Find the determinant of the matrix and check if it’s singular or nonsingular:

\[
\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \quad \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & 7 \\ -2 & -7 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}
\]
Introduction to Determinants

In order to define the determinant of an $n \times n$ matrix, we need some notation first:

$A_{ij}$ is the matrix obtained from matrix $A$ by deleting the $i$-th row and $j$-th column of $A$, while $a_{ij}$ is the entry located in the intersection of the $i$-th row and $j$-th column of $A$.

**Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad A_{23} = \begin{bmatrix} \end{bmatrix}$$
Let $A = [a_{ij}]$ be $n \times n$ matrix, with $n \geq 2$. Then:

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$
Determinants: Example

Find the determinant of $A$:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$
Determinants: Cofactor Expansion

Definition

Let \( A = [a_{ij}] \) be \( n \times n \) matrix, with \( n \geq 2 \). Then:

\[
\text{det} \ A = \sum_{j=1}^{n} (-1)^{1+j} \ a_{1j} \ \text{det} \ A_{1j}
\]

Definition

The \((i, j)\)-cofactor of \( A \) is an entry of \( n \times n \) matrix \( C_A \) where

\[
(C_A)_{ij} = (-1)^{i+j} \ \text{det} \ A_{ij}.
\]

Example: cofactor expansion across row 1

\[
\begin{vmatrix}
1 & 2 & 0 \\
3 & -1 & 2 \\
2 & 0 & 1 \\
\end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}
\]