MATH 2331 - 19859

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Eigenvectors & Eigenvalues

An eigenvector of an $n \times n$ matrix $A$ is a nonzero vector $x$ such that

$$Ax = \lambda x$$

for some scalar $\lambda$.

A scalar $\lambda$ is called an eigenvalue of $A$ if there is a nontrivial solution $x$ of $Ax = \lambda x$; such an $x$ is called an eigenvector corresponding to $\lambda$. 
Example

Show that 4 is an eigenvalue of \( A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \) and find the corresponding eigenvectors.
Example (cont.)
Eigenvectors & Eigenvalues: Example (cont.)

Eigenspace for $\lambda = 4$

Eigenspace

The set of all solutions to $(A - \lambda I)x = 0$ is called the **eigenspace** of $A$ corresponding to $\lambda$.

Warning

The method just used to find eigenvectors *cannot* be used to find eigenvalues.
Example

Let

\[ A = \begin{bmatrix}
2 & 0 & 0 \\
-1 & 3 & 1 \\
-1 & 1 & 3 \\
\end{bmatrix}. \]

An eigenvalue of \( A \) is \( \lambda = 2 \). Find a basis for the corresponding eigenspace.
Example (cont.)
Eigenspace: Example (cont.)

Effect of multiplying by $A$ the vectors in eigenspace associated to $\lambda = 2$: 

![Diagrams showing the effect of multiplying vectors by $A$ in the eigenspace associated to $\lambda = 2$.]
Suppose $\lambda$ is eigenvalue of $A$. Determine an eigenvalue of $A^2$ and $A^3$. In general, what is an eigenvalue of $A^n$?
In general, ______ is an eigenvalue of $A^n$. 
Eigenvectors and Linear Independence

**Theorem**

If \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) are eigenvectors that correspond to distinct eigenvalues \( \lambda_1, \ldots, \lambda_r \) of an \( n \times n \) matrix \( A \), then \( \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \) is a linearly independent set.
Example

Mark each statement as True or False and justify your answer.

1. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then $\lambda$ is an eigenvalue of $A$.

2. The eigenvalues of a matrix are on its main diagonal.
3. A number $c$ is an eigenvalue of $A$ if and only if the equation $(A - cl)x = 0$ has a nontrivial solution.

4. If $Ax = \lambda x$ for some scalar $\lambda$, the vector $x$ is an eigenvector of $A$. 
The Characteristic Equation

Section 5.2
The Characteristic Equation

We have seen that to find the eigenvectors \( \mathbf{x} \) of matrix \( A \), that is the nonzero vectors \( \mathbf{x} \) such that

\[
A\mathbf{x} = \lambda\mathbf{x},
\]

we need to solve \((A - \lambda I)\mathbf{x} = \mathbf{0}\) for a given eigenvalue \( \lambda \).

**Question**: How do we find the eigenvalues \( \lambda \) of \( A \)?
The Characteristic Equation

\[ x \text{ must be nonzero} \]
\[ \Downarrow \]
\[ (A - \lambda I) x = 0 \text{ must have nontrivial solutions} \]
\[ \Downarrow \]
\[ (A - \lambda I) \text{ is not invertible} \]
\[ \Downarrow \]
\[ \det (A - \lambda I) = 0 \]

Definitions

This last equation is called \textit{characteristic equation} and \( \det (A - \lambda I) \) is called \textit{characteristic polynomial}

Solve \( \det (A - \lambda I) = 0 \) for \( \lambda \) to find the eigenvalues.
Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$. 
Theorem

The eigenvalues of a triangular matrix are the diagonal entries.

Proof for the $3 \times 3$ Upper Triangular Case: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$  Then

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$
The Matlab command to compute the eigenvalues of matrix $A$ is:

```
eig(A)
```

**Exercise:** Verify with Matlab the answer to the exercise on slide 21.

If we need to find the eigenvectors, we can use the command in this way:

```
[V D] = eig(A)
```

where $D$ is a diagonal matrix with the eigenvalues on the diagonal, and the columns of matrix $V$ are the corresponding eigenvectors.
Example

Find the eigenvalues of

\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}. \]

Verify your answer with Matlab.
Example (cont.)
The Invertible Matrix Theorem - continued

Theorem (The Invertible Matrix Theorem - continued)

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

- The number 0 is not an eigenvalue of $A$.
- $\det A \neq 0$

Algebraic Multiplicity

The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation.
Example

Find the eigenvalues of $A$ and give their multiplicity:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
Example

Let $\lambda$ be an eigenvalue of an invertible matrix $A$. Show that $\lambda^{-1}$ is an eigenvalue of $A^{-1}$. 
Chapter 5

Sec. 5.1 Sec. 5.2

Eigenvectors and Linear Independence

Eigenspace

The set of all solutions to \((A - \lambda I)x = 0\) is called the eigenspace of \(A\) corresponding to \(\lambda\).

Theorem

If \(v_1, \ldots, v_r\) are eigenvectors that correspond to distinct eigenvalues \(\lambda_1, \ldots, \lambda_r\) of an \(n \times n\) matrix \(A\), then \(\{v_1, \ldots, v_r\}\) is a linearly independent set.