MATH 2331 - 17571

Vladimir Yushutin
yushutin@math.uh.edu
Office: PGH 606
Lecture: TuTh 1:00PM-2:30PM in SEC 203
Office hours: TuTh 2.30PM-4PM and BY APPOINTMENT
The Characteristic Equation

Section 5.2
We have seen that to find the eigenvectors $\mathbf{x}$ of matrix $A$, that is the nonzero vectors $\mathbf{x}$ such that

$$A \mathbf{x} = \lambda \mathbf{x},$$

we need to solve $(A - \lambda I) \mathbf{x} = \mathbf{0}$ for a given eigenvalue $\lambda$.

**Question:** How do we find the eigenvalues $\lambda$ of $A$?
The Characteristic Equation

\[ \mathbf{x} \text{ must be nonzero} \]
\[ \Downarrow \]
\[ (A - \lambda I) \mathbf{x} = \mathbf{0} \text{ must have nontrivial solutions} \]
\[ \Downarrow \]
\[ (A - \lambda I) \text{ is not invertible} \]
\[ \Downarrow \]
\[ \det (A - \lambda I) = 0 \]

**Definitions**

This last equation is called *characteristic equation* and \( \det (A - \lambda I) \) is called *characteristic polynomial*.

Solve \( \det (A - \lambda I) = 0 \) for \( \lambda \) to find the eigenvalues.
Example

Find the eigenvalues of \( A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \).
Theorem

The eigenvalues of a triangular matrix are the diagonal entries.

Proof for the $3 \times 3$ Upper Triangular Case: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$ Then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$
The Matlab command to compute the eigenvalues of matrix $A$ is:

```
eig(A)
```

If we need to find the eigenvectors, we can use the command in this way:

```
[V D] = eig(A)
```

where $D$ is a diagonal matrix with the eigenvalues on the diagonal, and the columns of matrix $V$ are the corresponding eigenvectors.
Example

Find the eigenvalues of

\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}. \]
The Invertible Matrix Theorem - continued

Theorem (The Invertible Matrix Theorem - continued)

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

- $\text{det } A \neq 0$
- The number 0 is not an eigenvalue of $A$.

Algebraic Multiplicity

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.
Example

Find the eigenvalues of $A$ and give their multiplicity:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
Theorem

Let \( \lambda \) be an eigenvalue of an invertible matrix \( A \). Show that \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \).
Chapter 5  Sec. 5.2

The Characteristic Equation

Definitions

Equation \( \det (A - \lambda I) = 0 \) is called characteristic equation and \( \det (A - \lambda I) \) is called characteristic polynomial.

Solve \( \det (A - \lambda I) = 0 \) for \( \lambda \) to find the eigenvalues.

Theorem

The eigenvalues of a triangular matrix are the diagonal entries.
Example

Find the eigenvalues of $A$ and give their multiplicity:

$$A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
5 & 3 & 0 & 0 \\
9 & 1 & 3 & 0 \\
1 & 2 & 5 & -1
\end{bmatrix}.$$
For $n \times n$ matrices $A$ and $B$, we say the $A$ is similar to $B$ if there is an invertible matrix $P$ such that

$$P^{-1}AP = B$$

or equivalently,

$$A = PBP^{-1}.$$

**Theorem**

If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial, and hence, the same set of eigenvalues.