The Characteristic Equation

Definitions
Equation \( \det (A - \lambda I) = 0 \) is called \textit{characteristic equation} and \( \det (A - \lambda I) \) is called \textit{characteristic polynomial}.

Solve \( \det (A - \lambda I) = 0 \) for \( \lambda \) to find the eigenvalues.

Theorem
\textit{The eigenvalues of a triangular matrix are the diagonal entries.}
The Invertible Matrix Theorem - continued

Theorem (The Invertible Matrix Theorem - continued)

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

- The number 0 is not an eigenvalue of $A$.
- $\det A \neq 0$

Algebraic Multiplicity

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.
Example

Find the eigenvalues of $A$ and give their multiplicity:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{bmatrix}.$$
Similarity

For $n \times n$ matrices $A$ and $B$, we say the $A$ is similar to $B$ if there is an invertible matrix $P$ such that

$$P^{-1}AP = B \quad \text{or equivalently,} \quad A = PBP^{-1}.$$ 

Theorem

If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues.
Diagonalization

Section 5.3
Diagonalization

**Goal:** Given an $n \times n$ matrix $A$, find an easy way to compute $A^k$ quickly for large $k$ by using a similarity relationship $A = PDP^{-1}$.

**Example:** Let

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$ 

Find $D^2$ and $D^3$. In general, what is $D^k$, where $k$ is a positive integer?
Powers of Diagonal Matrix

$D^k$ is trivial to calculate: it is the diagonal matrix that on the main diagonal has the diagonal entries of $D$ to the $k$-th power.
Example

Find a formula for $A^k$ given that $A = PDP^{-1}$ where $D$ is a diagonal matrix.
A square matrix $A$ is said to be **diagonalizable** if $A$ is similar to a diagonal matrix, i.e. if $A = PDP^{-1}$ where $P$ is invertible and $D$ is a diagonal matrix.

When is $A$ diagonalizable??

The answer lies in examining the eigenvalues and eigenvectors of $A$. 
Diagonalizable

In general, 

\[ A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \]

and if \( \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \) is invertible, \( A \) equals 

\[ \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1} \]
Diagonalization Theorem

Theorem

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are $n$ linearly independent eigenvectors of $A$. In this case, the diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$. 
Example

Diagonalize the following matrix, if possible.

\[ A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \]

Step 1. Find the eigenvalues of A.
Example (cont.)
Example (cont.)

Step 2. Find the eigenvectors of A.
Example (cont.)
Example (cont.)
Diagonalization: Example (cont.)

Step 3: Construct $P$ from the vectors at step 2 and check if it is invertible.

Step 4: Construct $D$ from the corresponding eigenvalues.
Diagonalization: Example (cont.)

Step 5: Check your work by verifying that $AP = PD$

$$AP = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$PD = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} =$$
Diagonalization

Theorem
An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

Example: Is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

diagonalizable? Why?
Example

Diagonalize the following matrix, if possible.

\[
A = \begin{bmatrix}
2 & 4 & 6 \\
0 & 2 & 2 \\
0 & 0 & 4
\end{bmatrix}.
\]
Example (cont.)
Example (cont.)
Example (cont.)
Example

Use Matlab to verify that $A$ is not diagonalizable:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$
Matlab exercise

Use Matlab to check if $A$ is diagonalizable:

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 24 & -12 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$
A, P, and D are $n \times n$ matrices. Mark each statement True or False. Justify your answer.

- A is diagonalizable if $A = PDP^{-1}$ for some matrix $D$ and some invertible matrix $P$.

- If $\mathbb{R}^n$ has a basis of eigenvectors of $A$, then $A$ is diagonalizable.
Example

$A$, $P$, and $D$ are $n \times n$ matrices. Mark each statement True or False. Justify your answer.

- $A$ is diagonalizable if and only if $A$ has $n$ eigenvalues, counting multiplicities.

- If $A$ is diagonalizable, then $A$ is invertible.