MATH 2331 - 17571

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Orthonormal Sets

A set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) in \( \mathbb{R}^n \) is called an \textbf{orthonormal set} if \( \mathbf{u}_i \cdot \mathbf{u}_j = 0 \) for \( i \neq j \) and \( \mathbf{u}_i \cdot \mathbf{u}_j = 1 \) for \( i = j \).

Example: Is \( \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) an orthonormal set?
Example

Is \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) an orthonormal set?
Orthonormal Sets

Orthonormal Basis
Suppose \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthonormal set of nonzero vectors in \( \mathbb{R}^n \) and \( W = \text{Span}\{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \), then \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthonormal basis for \( W \).

Suppose \( U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] \) where \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is an orthonormal set.

\[
U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = 
\]
Orthonormal Matrix: Theorems

**Theorem**

An $m \times n$ matrix $U$ has orthonormal columns if and only if $U^T U = I$.

**Theorem**

Let $U$ be an $m \times n$ matrix with orthonormal columns, and let $x$ and $y$ be in $\mathbb{R}^n$. Then

a. $\|Ux\| = \|x\|

b. $(Ux) \cdot (Uy) = x \cdot y$

c. $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$. 
Orthonormal Matrix: Example

Orthogonal Matrix

If the columns of $U$ are $n$ orthonormal vectors in $\mathbb{R}^n$, then:

$$U^T U = UU^T = I.$$ 

So

$$U^{-1} = U^T$$

Such a matrix is called **orthogonal matrix**.
Describe all linear transformations of $\mathbb{R}^2$ with orthonormal standard matrices.
Orthogonal Projections

Section 6.3
Orthogonal Projections

Given two nonzero vectors $\mathbf{y}$ and $\mathbf{u}$ in $\mathbb{R}^n$, suppose we want to write $\mathbf{y}$ in the following way:

$$\mathbf{y} = (\text{multiple of } \mathbf{u}) + (\text{a vector } \perp \text{ to } \mathbf{u}) = \alpha \mathbf{u} + \mathbf{u}^\perp$$
Orthogonal Projections

\[ u \perp \cdot u = 0 \implies (y - \alpha u) \cdot u = 0 \implies y \cdot u - \alpha (u \cdot u) = 0 \]

\[ \implies \alpha = \frac{y \cdot u}{u \cdot u} \]

\[ \hat{y} = \alpha u = \frac{y \cdot u}{u \cdot u} \]

is the orthogonal projection of \( y \) onto \( u \)
Orthogonal Projections

\[ \hat{y} \perp = y - \hat{y} = y - \alpha y = y - \frac{y \cdot u}{u \cdot u} u \]

The distance between the tip of \( y \) and the line passing through \( u \) is given by \( \| \hat{y} \perp \| \).
Example

Given:

\[ y = \begin{bmatrix} -8 \\ 4 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \]

Find the orthogonal projection of \( y \) onto \( u \) and the distance between the tip of vector \( y \) and the line through \( u \).
\[ \hat{y} = \frac{y \cdot u}{u \cdot u} u \] is the **orthogonal projection of** \( y \) **onto** \( u \).
Orthogonal Projection

**Theorem**
Suppose \( \{u_1, \ldots, u_p\} \) is an orthogonal basis for \( W \) in \( \mathbb{R}^n \). For each \( y \) in \( W \), we have:

\[
y = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_p}{u_p \cdot u_p} \right) u_p
\]

**Proof:**
Example

Suppose \( \{u_1, u_2, u_3\} \) is an orthogonal basis for \( \mathbb{R}^3 \) and let \( W = \text{Span}\{u_1, u_2\} \). Write \( y \) in \( \mathbb{R}^3 \) as the sum of a vector in \( W \) and a vector in \( W^\perp \).
The Orthogonal Decomposition Theorem

**Theorem**

Let $W$ be a subspace of $\mathbb{R}^n$. Then each $y$ in $\mathbb{R}^n$ can be uniquely represented in the form

$$ y = \hat{y} + \hat{y}^\perp $$

where $\hat{y}$ is in $W$ and $\hat{y}^\perp$ is in $W^\perp$. In fact, if $\{u_1, \ldots, u_p\}$ is any orthogonal basis of $W$, then

$$ \hat{y} = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_p}{u_p \cdot u_p} \right) u_p $$

and

$$ \hat{y}^\perp = y - \hat{y}. $$

The vector $\hat{y}$ is called the **orthogonal projection of $y$ onto $W$**.
The Orthogonal Decomposition Theorem

\[ \hat{y} = \text{proj}_W y \]
In $\mathbb{R}^3$, define vectors:

$$
y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.
$$

Find the orthogonal projection of $y$ onto $\text{Span}\{u_1, u_2\}$.
In $\mathbb{R}^4$, define vectors:

$$
\mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.
$$

Find the orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
Orthogonal Matrix

If $U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}$. Then $U^T = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_p^T \end{bmatrix}$.

So

$$UU^T = u_1u_1^T + u_2u_2^T + \cdots + u_pu_p^T$$

and

$$\left(UU^T\right)y = \left(u_1u_1^T + u_2u_2^T + \cdots + u_pu_p^T\right)y$$
Orthogonal Projection: Theorem

**Theorem**

- If \( \{u_1, \ldots, u_p\} \) is an orthonormal basis for a subspace \( W \) of \( \mathbb{R}^n \), then
  \[
  \hat{y} = (y \cdot u_1) u_1 + \cdots + (y \cdot u_p) u_p
  \]

- If \( U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} \), then
  \[
  \hat{y} = UU^T y \quad \text{for all } y \text{ in } \mathbb{R}^n.
  \]

**Outline of Proof:**

\[
\hat{y} = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_p}{u_p \cdot u_p} \right) u_p
\]

\[
= (y \cdot u_1) u_1 + \cdots + (y \cdot u_p) u_p = UU^T y.
\]