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Lecture : MoWeFr 10:00AM-11:00AM in CBB 104

Office hours : Fr 3:00PM-4:00PM and BY APPOINTMENT
Example

Given

\[ W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}, \]

find \( W^\perp \).
Example

Given $W = \text{Span}(v_1, ..., v_k), v_1, ..., v_k \in \mathbb{R}^n$,

what is the dimension of $W^\perp$?
Theorem

Let $A$ be an $m \times n$ matrix. Then the orthogonal complement of the column space of $A$ is the nullspace of $A^T$:

$$(Col A)^\perp = \text{Nul } A^T.$$
Example

Let

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}. \]

Find Col \( A \) and Nul \( A^T \) and check that \((\text{Col } A)^\perp = \text{Nul } A^T\).
Orthogonal Sets

Section 6.2
Orthogonal Sets

Orthogonal

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Orthogonal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is called an orthogonal set if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

Example: Is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set?
Example

Is $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set?
Orthogonal Sets: Theorem

Theorem

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$. Then $S$ is a linearly independent set and is therefore a basis for $W$.

Proof:
An **orthogonal basis** for a subspace $W$ of $\mathbb{R}^n$ is a basis for $W$ that is also an orthogonal set.

Suppose $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ and suppose $y$ is in $W$. Find $c_1, \ldots, c_p$ so that

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p.$$
Orthogonal Basis: Theorem

**Theorem**

Let \( \{u_1, u_2, \ldots, u_p\} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \). Then each \( y \) in \( W \) has a unique representation as a linear combination of \( u_1, u_2, \ldots, u_p \). In fact, if

\[
y = c_1u_1 + c_2u_2 + \cdots + c_pu_p
\]

then

\[
c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad (j = 1, \ldots, p)
\]
Example

Express

\[ y = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \]

as a linear combination of the orthogonal basis:

\[ \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \]
A set of vectors \( \{u_1, u_2, \ldots, u_p\} \) in \( \mathbb{R}^n \) is called an **orthonormal set** if \( u_i \cdot u_j = 0 \) for \( i \neq j \) and \( u_i \cdot u_j = 1 \) for \( i = j \).

**Example:** Is \( \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) an orthonormal set?
Example

Is \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) an orthonormal set?