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Lecture: MoWeFr 10:00AM-11:00AM in CBB 104
Office hours: Fr 3:00PM-4:00PM and BY APPOINTMENT
Orthonormal Sets

A set of vectors \( \{u_1, u_2, \ldots, u_p\} \) in \( \mathbb{R}^n \) is called an **orthonormal set** if \( u_i \cdot u_j = 0 \) for \( i \neq j \) and \( u_i \cdot u_j = 1 \) for \( i = j \).

**Theorem**

Suppose \( \{u_1, \ldots, u_p\} \) is an orthogonal basis for \( W \) in \( \mathbb{R}^n \). For each \( y \) in \( W \), we have:

\[
y = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_p}{u_p \cdot u_p} \right) u_p
\]
Orthogonal Projections

Section 6.3
Orthogonal Projections

Given two nonzero vectors $y$ and $u$ in $\mathbb{R}^n$, suppose we want to write $y$ in the following way:

$$y = (\text{multiple of } u) + (\text{a vector } \perp \text{ to } u) = \hat{y} + y_{\perp}$$
Orthogonal Projections

\[ y^\perp \cdot u = 0 \quad \implies \quad (y - \alpha u) \cdot u = 0 \quad \implies \quad y \cdot u - \alpha (u \cdot u) = 0 \]

\[ \implies \quad \alpha = \frac{y \cdot u}{u \cdot u} \]

\[ \hat{y} = \alpha u = \frac{y \cdot u}{u \cdot u} \quad \text{is the orthogonal projection of } y \text{ onto } u \]
Orthogonal Projections

\[ y \perp = y - \hat{y} = y - \alpha y = y - \frac{y \cdot u}{u \cdot u} u \]

The distance between the tip of \( y \) and the line passing through \( u \) is given by \( \| \hat{y} \perp \| \)
Example

Given:

\[ y = \begin{bmatrix} -8 \\ 4 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \]

Find the orthogonal projection of \( y \) onto \( u \) and the distance between the tip of vector \( y \) and the line through \( u \).
Orthogonal Projection

\[ \hat{y} = \frac{y \cdot u}{u \cdot u} u \text{ is the orthogonal projection of } y \text{ onto } u. \]

\[ y^\perp = y - \frac{y \cdot u}{u \cdot u} u \text{ is orthogonal to } u. \]
Example

Suppose \( \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \) is an orthogonal basis for \( \mathbb{R}^3 \) and let \( W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \). Write \( \mathbf{y} \) in \( \mathbb{R}^3 \) as the sum of a vector in \( W \) and a vector in \( W^\perp \).
The Orthogonal Decomposition Theorem

Theorem

Let \( W \) be a subspace of \( \mathbb{R}^n \). Then each \( y \) in \( \mathbb{R}^n \) can be uniquely represented in the form

\[
y = \hat{y} + \hat{y}^\perp
\]

where \( \hat{y} \) is in \( W \) and \( \hat{y}^\perp \) is in \( W^\perp \). In fact, if \( \{u_1, \ldots, u_p\} \) is any orthogonal basis of \( W \), then

\[
\hat{y} = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_p}{u_p \cdot u_p} \right) u_p
\]

and

\[
\hat{y}^\perp = y - \hat{y}.
\]

The vector \( \hat{y} \) is called the **orthogonal projection of** \( y \) **onto** \( W \).
The Orthogonal Decomposition Theorem

\[ \hat{y} = \text{proj}_W y \]
Example

In $\mathbb{R}^3$, define vectors:

\[ y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \]

Find the orthogonal projection of $y$ onto $\text{Span}\{u_1, u_2\}$. 

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In $\mathbb{R}^4$, define vectors:

\[ y = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}. \]

Find the orthogonal projection of $y$ onto $\text{Span}\{u_1, u_2, u_3\}$.