MATH 2331 - 20947

Vladimir Yushutin
yushutin@math.uh.edu
Office : PGH 606
Lecture : MoWeFr 10:00AM-11:00AM in CBB 104
Office hours : Fr 3:00PM-4:00PM and BY APPOINTMENT
Consider $U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}$, the standard matrix of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(x) = Ux$.

What is the projection onto $\text{Col } U = \text{Span}(u_1, u_2, \cdots, u_m)$ where $\{u_1, u_2, \cdots, u_m\}$ is an orthonormal set?
\( \text{proj}_W y \) is the projection onto \( W = \text{Col} \ U = \text{Span}(u_1, u_2, \cdots, u_m) \)

\( y - \text{proj}_W y \) belongs to \( W^\perp = (\text{Col} \ U)^\perp = \text{Nul} \ U^T \)
Suppose \( U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] \) where \( \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \) is an orthonormal set in \( \mathbb{R}^n \).

\[
U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = I_{3 \times 3}
\]

\[
UU^T = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} = ?_{n \times n}
\]
Chapter 6

Orthonormal Matrix: Theorems

Theorem

An $m \times n$ matrix $U$ has orthonormal columns if and only if $U^T U = I$.

Theorem

Let $U$ be an $m \times n$ matrix with orthonormal columns, and let $x$ and $y$ be in $\mathbb{R}^n$. Then

a. $\|Ux\| = \|x\|$ (lengths are conserved by $x \rightarrow Ux$)

b. $(Ux) \cdot (Uy) = x \cdot y$ (angles are conserved by $x \rightarrow Ux$)
Orthogonal Matrix

If the columns of $U$ are $n$ orthonormal vectors in $\mathbb{R}^n$, then:

$$U^T U = UU^T = I.$$ 

So

$$U^{-1} = U^T$$

Such a matrix is called orthogonal matrix.

Columns of an orthogonal matrix form an orthonormal basis for $\mathbb{R}^n$.
Orthogonal Matrix

If \( U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} \). Then \( U^T = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_p^T \end{bmatrix} \).

So

\[
UU^T = u_1u_1^T + u_2u_2^T + \cdots + u_pu_p^T
\]

and

\[
\left( UU^T \right)y = \left( u_1u_1^T + u_2u_2^T + \cdots + u_pu_p^T \right)y
\]
Orthogonal Projection: Theorem

**Theorem**

- If \( \{u_1, \ldots, u_p\} \) is an orthonormal basis for a subspace \( W \) of \( \mathbb{R}^n \), then
  \[
  \hat{y} = (y \cdot u_1) u_1 + \cdots + (y \cdot u_p) u_p
  \]

- If \( U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} \), then
  \[
  \hat{y} = UU^T y \quad \text{for all } y \in \mathbb{R}^n.
  \]

**Outline of Proof:**

\[
\hat{y} = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_p}{u_p \cdot u_p} \right) u_p
\]

\[
= (y \cdot u_1) u_1 + \cdots + (y \cdot u_p) u_p = UU^T y.
\]
Example

In $\mathbb{R}^3$, define vectors:

\[ y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \]

Find the orthogonal projection of $y$ onto $\text{Span}\{u_1, u_2\}$.
Example

Describe all linear transformations of $\mathbb{R}^2$ with orthonormal standard matrices.