MATH 2331 - 20947

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Lecture: MoWeFr 10:00AM-11:00AM in CBB 104
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Singular Value Decomposition

Section 7.3*
Motivation for SVD

Consider a nondiagonalizable or even a nonsquare matrix. Do we give up on it?
No!

Let’s consider an alternative decomposition: \( A = UDV^{-1} \), where \( U \) and \( V \) may be different.

Hint#1: find vectors \( u_k \) and \( v_k \) such that

\[
Av_k = \sigma_k u_k
\]

Hint#2: the largest eigenvalue of a square matrix, \( \lambda_{\text{max}} \), if it exists, has the highest output-to-input ratio:

\[
\max \frac{\|Ax\|}{\|x\|} = \lambda_{\text{max}}
\]
Singular Value Decomposition

SVD

Let \( A \) be an \( m \times n \) matrix then there is an orthogonal \( m \times m \) matrix \( V \) and an orthogonal \( n \times n \) matrix \( U \) such that

\[
A = U \Sigma V^T,
\]

where

\[
\Sigma = \begin{bmatrix}
\sigma_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \sigma_2 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ddots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \sigma_r & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

Diagonal elements of \( \Sigma \) are called **singular values** and they are positive.
Example

Let $Q$ be a rotation matrix: $Q^T = Q^{-1}$.
Find the singular value decomposition of $Q$. 
Example

Let $A = U \Sigma V^T$ be the SVD of a square matrix $A$. Find the polar decomposition $UP$ of the matrix $A$, where

$U$ is orthogonal

$P$ is symmetric and all the eigenvalues of $P$ are nonnegative.
How to find SVD

Given an SVD of $A$, compute

$$A^T A = (V^{TT} \Sigma U^T)(U \Sigma V^T) = V^{TT} \Sigma I \Sigma V^T = V \Sigma^2 V^T = V \Sigma^2 V^{-1}$$

$$A A^T = (U \Sigma V^T)(V^{TT} \Sigma U^T) = U \Sigma I \Sigma U^T = U \Sigma^2 U^T = U \Sigma^2 U^{-1}$$

We see that

$A^T A$ is diagonal in the basis of columns of $V$

$A A^T$ is diagonal in the basis of columns of $U$

Note: $A^T A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $A A^T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are both symmetric matrices, no surprise that they are diagonalizable.
How to find SVD

Key steps:

- compute eigenvalues $\lambda_k = \sigma_k^2$ of $A^T A$
- form $\Sigma$ with $\sigma_k = \sqrt{\lambda_k}$
- diagonalize $A^T A$ to get an orthogonal basis of columns of $V$
- derive matrix $U$ from $AV = U\Sigma$
Example

Let a shear transformation of $\mathbb{R}^3$ is given by

$$A = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ 

Find an singular value decomposition of the $A$. 