Exam 1 – Advanced Linear Algebra I (MATH 4377/6308) — Fall 2013

All exams will be collected at 12:45 PM sharp!

- Show all your work. If you use additional sheets of paper, turn these in too.
- If you want to use a theorem, state the theorem to the best of your ability.
- No calculators. No notes. No books. No use of phones during the exam.
- Any student caught cheating will receive a zero and reported via the university’s Academic Dishonesty policies (Article 5 of the student handbook).

1. (a) Write down mathematically what it means for a set of vectors \( v_1, \ldots, v_n \) to be linearly independent.
   (b) Write down mathematically what it means for a set of vectors \( v_1, \ldots, v_n \) to be linearly dependent.
   (c) Write down three vectors \( v_1, v_2, v_3 \in \mathbb{R}^3 \) that are linearly dependent.
2. Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ -x_1 + 2x_2 \end{pmatrix}$$

(a) Show that $T$ is a linear map.
(b) Is $T$ injective or not? Show or explain why.
(c) Is $T$ surjective or not? Show or explain why.
(d) Is $T$ invertible or not? Explain why.
(e) Find $\dim N(T)$ and $\dim R(T)$.
(f) Find the matrix $M(T)$ that encodes $T$ using $\{(1, 1)^T; (1, -1)^T\}$ as the domain space’s basis and $\{(1, 0)^T; (0, 1)^T\}$ as the target space’s basis.
3. Let \( X = \mathbb{R}^3 \) and define

\[
P = \begin{pmatrix}
\frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{1}{6} & \frac{1}{3} & \frac{5}{6}
\end{pmatrix}
\]

(a) What is the definition of a projection?
(b) Show that \( P \) is a projection.
(c) Compute \( N(P) \) and \( \dim N(P) \).
(d) Show that \( I - P \) is a projection.
(e) Compute \( N(I - P) \) and \( \dim N(I - P) \).
(f) Prove that if a matrix \( Q \) is similar to \( P \), then \( I - Q \) is similar to \( I - P \).
4. Let $V$ be the subspace of $\mathbb{R}^3$ spanned by $v_1 = (1, 0, -1)$ and $v_2 = (0, 1, 0)$:
(a) Find a set of linear functionals $l(v)$ that span the annihilator $V^\perp$ of $V$.
(b) List a set of vectors that span $V^\perp\perp$, the annihilator of $V^\perp$.
(c) For a finite-dimensional space $X \equiv \mathbb{R}^n$ ($n$ finite), what is the annihilator $X^\perp$ of $X$?
5. (a) Define $X = \mathbb{R}^3$. Define $Y$ as the subspace with a single basis vector $(1, 0, 0)^T$. Define $Z$ as the subspace with the single basis vector $(0, 1, 0)^T$. Prove that the union of $Y$ and $Z$ ($Y \cup Z$), the combined set of all elements of $Y$ and all elements of $Z$, is not a subspace of $X$.

(b) Say $Y$ and $Z$ are linear subspaces of $X$, and they are non-overlapping, i.e. their intersection $Y \cap Z = \{0\}$. Prove that their union $Y \cup Z$ is not a subspace of $X$. 

