Homework 1 – Advanced Linear Algebra I (MATH 4377/6308) — Fall 2013

DUE THURSDAY SEPTEMBER 5 AT 11:30AM. HAND IN AT LECTURE.

Start early! You may work with others on this assignment, but the solutions you write out to these problems must be your own. Do not copy others’ solutions! Please show all of the steps necessary to complete the written problem.

1. Read Chapter 1 of Lax: Fundamentals (pp.1-12)

2. (a) Let $0 \in S$ be the identity element of the linear space $S$. Prove that $\forall a \in \mathbb{R}, a0 = 0$.
   
   **Proof:** Take any $a \in \mathbb{R}$, then using the definition of the identity element
   \[ a0 = a(0 + 0) = a0 + a0. \]
   By adding the additive inverse of $a0$ to the left and right hand sides of the above equation, we have
   \[ 0 = a0. \]

   (b) Show that the set of vectors $\{x_1, x_2, x_3, 0\}$, where 0 is the zero vector, is always linearly dependent.
   **Proof:** For any $a \in \mathbb{R}$, we can always write
   \[ 0x_1 + 0x_2 + 0x_3 + a0 = a0 = 0, \]
   where the last equality is due to the result from problem 2. Thus, there is always a linear combination of the set of vectors that sums to the identity element, so the set is linearly dependent.

3. Prove that if $Y$ and $Z$ are linear subspaces of $X$, their intersection $Y \cap Z$ is also a subspace of $X$.
   **Proof:** By definition of $W = Y \cap Z$, any $w \in W$ is in $Y$ and $Z$.
   
   **Closure under addition:** Take any pair $w_1, w_2 \in W$ then $w_1, w_2 \in Y$, so $w_1 + w_2 \in Y$; also $w_1, w_2 \in Z$, so $w_1 + w_2 \in Z$. Therefore any $w_1, w_2 \in W$ satisfies $w_1 + w_2 \in W$.
   
   **Closure under scalar multiplication:** Take any $w \in W$ then $w \in Y$, so $aw \in Y$ for any $a \in \mathbb{R}$; also $w \in Z$, so $aw \in Z$ for any $a \in \mathbb{R}$. Therefore any $w \in W$ satisfies $aw \in W$ for any $a \in \mathbb{R}$.

4. State which of these sets of vectors $x = (x_1, ..., x_n)$ in $\mathbb{R}^n$ are a subspace of $\mathbb{R}^n$ and prove why.
   (a) $S$ is the set of all $x$ such that $x_1 \geq 0$.
   **Not a subspace.** Take an element $x = (x_1, ..., x_n)$ such that $x_1 > 0$, then by taking a scalar $a \in \mathbb{R}$ such that $a < 0$, then $y = (y_1, ..., y_n) = ax = (ax_1, ..., ax_n)$, so $y_1 = ax_1 < 0$, so $y \notin S$, so $S$ is not a subspace of $\mathbb{R}^n$.

   (b) $S$ is the set of all $x$ such that $x_1 - x_2 = 0$.
   **Subspace.** Take any two elements $x, y \in S$, then $x_1 - x_2 = 0$, so $x_1 = x_2$; also $y_1 - y_2 = 0$, so $y_1 = y_2$. Therefore
   \[ w = x + y = (x_1, x_1, x_3, ..., x_n) + (y_1, y_1, y_3, ..., y_n) = (x_1 + y_1, x_1 + y_1, x_3 + y_3, ..., x_n + y_n), \]
so $w_1 = x_1 + y_1 = w_2$, so the set $S$ is closed under addition.

Also, $\forall x \in S$ and $\forall a \in \mathbb{R}$

$$w = ax = (ax_1, ax_1, ax_3, \ldots, ax_n),$$

then $w_1 = w_2$, so $w \in S$, so $S$ is closed under scalar multiplication.

Therefore, $S$ is a subspace of $\mathbb{R}^n$.

5. Let $U, V,$ and $W$ be subspaces of some finite dimensional space $X$. Prove

$$\dim(U + V + W) = \dim U + \dim V + \dim W - \dim(U \cap V) - \dim(U \cap W) - \dim(V \cap W) + \dim(U \cap V \cap W).$$

**Note: The above statement is not true.** Therefore, the statement cannot be proven. I will point out the place at which the proof fails.

**Proof:** First, we assign $Y = U + V$, so

$$\dim(Y + W) = \dim Y + \dim W - \dim(Y \cap W)$$

by Theorem 7 (in Lax). Furthermore, we can write

$$\dim Y = \dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

Next, note that

$$\dim(Y \cap W) = \dim((U + V) \cap W) = \dim((U \cap W) + (V \cap W))$$

The last equality cannot be done in general (sometimes it will be true, but there are situations where it will NOT be true). The intersection $\cap$ of a sum with another subspace is not necessarily equal to the sum of intersections. To see this, consider

$$U = \left\{ x \in \mathbb{R}^2 | x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \right\} \quad V = \left\{ x \in \mathbb{R}^2 | x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right\} \quad W = \left\{ x \in \mathbb{R}^2 | x = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \right\}$$

Then $(U + V) \cap W = W$ but $U \cap W + V \cap W = 0$. Free 5 points for all who attempted the problem. My mistake.

6. Let $u = (2, 1, 3)^T$, $v = (v_1, v_2, v_3)^T$, $w = (5, 3, 1)^T$.

(a) Compute $3u, 2v, u - v + 2w, 2w - v$.

$$3u = \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}, \quad 2v = \begin{pmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{pmatrix}, \quad u - v + 2w = \begin{pmatrix} 12 - v_1 \\ 7 - v_2 \\ 5 - v_3 \end{pmatrix}, \quad 2w - v = \begin{pmatrix} 10 - v_1 \\ 6 - v_2 \\ 2 - v_3 \end{pmatrix}$$

(b) Choose values $(v_1, v_2, v_3)$ that would make $u, v,$ and $w$ linearly dependent.
One solution is if we take $v_1 = 12$, $v_2 = 7$, and $v_3 = 5$, then

$$ u - v + 2w = \begin{pmatrix} 12 - 12 \\ 7 - 7 \\ 5 - 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0. $$

But any choice such that we can find a nontrivial linear combination $c_1u + c_2v + c_3w = 0$ works.

7. Consider $x_1 = (1, 2, 3); x_2 = (0, 2, 1); x_3 = (0, 1, -1)$. Write $v = (1, 1, 1)$ as a linear combination of $x_1, x_2,$ and $x_3$.

We want to write $v = c_1x_1 + c_2x_2 + c_3x_3$. Thus, we construct the linear system

$$
\begin{align*}
c_1 + 0 \cdot c_2 + 0 \cdot c_3 &= 1 \\
2c_1 + 2c_2 + c_3 &= 1 \\
3c_1 + c_2 - c_3 &= 1
\end{align*}
$$

The first equation implies $c_1 = 1$. Plugging this into the 2nd and 3rd equations

$$
\begin{align*}
2c_2 + c_3 &= -1 \\
c_2 - c_3 &= -2
\end{align*}
$$

Adding these two equations, we have $3c_2 = -3$, so $c_2 = -1$. Plugging this into the second equation above, we have

$$
-1 - c_3 = -2 \implies -c_3 = -1 \implies c_3 = 1,
$$

so

$$
1 \cdot x_1 - 1 \cdot x_2 + \cdot x_3 = \begin{pmatrix} 1 - 0 + 0 \\ 2 - 2 + 1 \\ 3 - 1 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = v
$$