Homework 2 – Advanced Linear Algebra I (MATH 4377/6308) — Fall 2013

Due Thursday September 12 at 11:30AM. Hand in at lecture.

Start early! You may work with others, but the solutions you hand in must be your own. Do not copy others’ solutions! Please show all of the steps necessary to complete the written problem.

1. Read (a) Chapter 3: Linear Mappings (pp.19-31) in Lax
(b) Chapter 6: Linear Maps (pp. 62-81) in Lankham, Nachtergaele, and Schilling

2. Prove that the following map satisfies the properties of a linear map.

\[
T \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} x_2 - 2x_1 \\ x_1 + 3x_2 \end{array} \right)
\]

Take any \( x, y \in \mathbb{R}^2 \), then

\[
T(x + y) = T \left( \begin{array}{c} x_1 + y_1 \\ x_2 + y_2 \end{array} \right) = \left( \begin{array}{c} x_2 + y_2 - 2x_1 - 2y_1 \\ x_1 + y_1 + 3x_2 + 3y_2 \end{array} \right)
\]

\[
= \left( \begin{array}{c} x_2 - 2x_1 \\ x_1 + 3x_2 \end{array} \right) + \left( \begin{array}{c} y_2 - 2y_1 \\ y_1 + 3y_2 \end{array} \right) = Tx + Ty
\]

and \( T \) satisfies additivity. Also, taking any \( x \in \mathbb{R}^2 \) and \( a \in \mathbb{R} \), then

\[
T(ax) = T \left( \begin{array}{c} ax_1 \\ ax_2 \end{array} \right) = \left( \begin{array}{c} ax_2 - 2ax_1 \\ ax_1 + 3ax_2 \end{array} \right)
\]

\[
= a \left( \begin{array}{c} x_2 - 2x_1 \\ x_1 + 3x_2 \end{array} \right) = aTx
\]

so \( T \) satisfies homogeneity, and thus \( T \) is a linear map.

3. Show that the following map does not satisfy the properties of a linear map.

\[
T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \left( \begin{array}{c} x_1 \\ x_2^2 \\ x_3^3 \end{array} \right)
\]

Take any \( x \in \mathbb{R}^3 \) and \( a \in \mathbb{R} \), then

\[
T(ax) = T \left( \begin{array}{c} ax_1 \\ ax_2 \\ ax_3 \end{array} \right) = \left( \begin{array}{c} ax_1 \\ a^2x_2^2 \\ a^3x_3^3 \end{array} \right) = a \left( \begin{array}{c} x_1 \\ ax_2^2 \\ a^3x_3^3 \end{array} \right) \neq aT(x) = a \left( \begin{array}{c} x_1 \\ x_2^2 \\ x_3^3 \end{array} \right)
\]

when \( a \neq 1 \). So \( T \) does not satisfy homogeneity, so it is not a linear map.
4. Let $T$ be the linear mapping $T(p(z)) = p'(z)$, which takes the first derivative of a function $p(z)$, and $S$ be the linear mapping $S(p(z)) = (z+1)p(z)$, also acting on a function $p(z)$. Apply the products of the linear mappings $T \circ S$ and $S \circ T$ to the functions (a) $p(z) = e^z$ and (b) $p(z) = z^2$.

(a) $(T \circ S)(e^z) = \frac{d}{dz} [(z+1)e^z] = e^z + (z+1)e^z$

(b) $(T \circ S)(z^2) = \frac{d}{dz} [z^3 + z^2] = 3z^2 + 2z$

5. Let $J$ be the integral mapping $J : \mathbb{F} \to \mathbb{R}$ (mapping functions $\mathbb{F}$ to real numbers $\mathbb{R}$) such that

$$J(f) = \int_{-1}^{1} f(t) \, dt$$

(a) Prove $J$ is additive ($J(f + g) = J(f) + J(g)$).

Take any $f, g \in \mathbb{F}$, then

$$J(f + g) = \int_{-1}^{1} f(t) + g(t) \, dt = \int_{-1}^{1} f(t) \, dt + \int_{-1}^{1} g(t) \, dt = J(f) + J(g)$$

(b) Prove $J$ is homogeneous ($J(af) = aJ(f)$).

Take any $f \in \mathbb{F}$ and $a \in \mathbb{R}$, then

$$J(af) = \int_{-1}^{1} af(t) \, dt = a \int_{-1}^{1} f(t) \, dt = aJ(f)$$

(c) Find an $f \in N(J)$ where $f \neq 0$.

If we consider $f(t) = t$, then

$$J(t) = \int_{-1}^{1} t \, dt = \left[ \frac{t^2}{2} \right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$$  

Any odd function $f(t)$ will work, since this means $F(1) = F(-1)$, where $F'(t) = f(t)$.

6. Show that the following linear map is bijective:

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + x_3 \\ 2x_1 + 2x_3 \\ x_2 + x_3 \end{pmatrix}$$

First, we prove $T$ is injective by showing $N(T) = 0$. To do so, we simply attempt to solve $Tx = 0$ using row operations

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This proves $x_1 = x_2 = x_3 = 0$ must be satisfied for a vector $x$ to be in the null space $N(T)$, so $T$ is injective.
Now, to show $T$ is surjective. To do so, we must show $R(T) = \mathbb{R}^3$, we must prove that $\forall z \in \mathbb{R}^3$, $\exists x \in \mathbb{R}^3$ such that $Tx = z$. To do so, we solve

\[
\begin{pmatrix}
1 & -1 & 1 & | & z_1 \\
2 & 0 & 2 & | & z_2 \\
0 & 1 & 1 & | & z_3
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & -1 & 1 & | & z_1 \\
0 & 2 & 0 & | & z_2 - 2z_1 \\
0 & 1 & 1 & | & z_3
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & -1 & 1 & | & z_1 \\
0 & 1 & 0 & | & z_2/2 - z_1 \\
0 & 0 & 1 & | & z_3 - z_2/2 + z_1
\end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix}
1 & 0 & 0 & | & -z_1 + z_2 - z_3 \\
0 & 1 & 0 & | & -z_1 + z_2/2 \\
0 & 0 & 1 & | & z_1 - z_2/2 + z_3
\end{pmatrix}
\]

Thus, we can obtain any $z \in \mathbb{R}^3$ using the linear map $Tx$ by taking $x_1 = -z_1 + z_2 - z_3$, $x_2 = -z_1 + z_2/2$, and $x_3 = z_1 - z_2/2 + z_3$. Thus, $T$ is surjective and injective, so it is bijective.

7. Suppose $T : X \rightarrow U$ is an injective linear map. Given a linearly independent list of vectors $x_1, x_2, \ldots, x_n \in X$, prove the list $T(x_1), T(x_2), \ldots, T(x_n) \in U$ is linearly independent.

Assume we have a linear combination $\sum_{j=1}^{n} c_j T(x_j) = 0$.

This means $T \left( \sum_{j=1}^{n} c_j x_j \right) = 0$, since $T$ is a linear map.

This then implies $\left( \sum_{j=1}^{n} c_j x_j \right) \in N(T)$.

Since $T$ is injective, only $0 \in N(T)$, which means $\sum_{j=1}^{n} c_j x_j = 0$.

Since $x_1, x_2, \ldots, x_n$ are linearly independent, it must be the case that $c_j = 0$ for all $j = 1, \ldots, n$.

This means $\sum_{j=1}^{n} c_j T(x_j) = 0$ implies all $c_j = 0$, so $T(x_1)$, $T(x_2)$, $\ldots$, $T(x_n)$ are linearly independent.