Start early! You may work with others, but the solutions you hand in must be your own. Do not copy others’ solutions! Please show all of the steps necessary to complete the written problem.

1. **Read** Chapter 2: Duality (pp.13-18) in Lax

2. Given a nonzero vector $v \in X$, show there is a linear function $l$ such that $l(v) \neq 0$.

3. Let $Y$ be a subspace of $X$. Prove that the annihilator $Y^\perp$ is a subspace of $X'$.

4. Let $V$ be the subspace of $\mathbb{R}^4$ spanned by $v_1 = (-2, 0, 1, 3)$ and $v_2(1, 1, 1, 1)$.
   (a) Which linear functionals $l(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator $V^\perp$ of $V$?
   (b) Show that $v_1$ and $v_2$ are in $V'^\perp$, the annihilator of $V'$.

5. Take the interval $w = [-1, 1]$ and $\mathbb{P}_3$ to be the space of all polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.
   Take the four points $-a, -b, b, a \in w$.
   (a) Determine the weights $m_1, m_2, m_3, m_4$ so that
   \[
   \int_w p(x)dx = m_1p(-a) + m_2p(-b) + m_3p(b) + m_4p(a)
   \]
   holds for all polynomials in $\mathbb{P}_3$.
   (b) For what values of $a$ and $b$ are the weights positive?

6. Show that, when $T, R \in \mathcal{L}(X, U)$ and $S \in \mathcal{L}(U, V)$:
   (a) $(ST)' = T'S'$;
   (b) $(T + R)' = T' + R'$;
   (c) $(T^{-1})' = (T')^{-1}$

7. (a) Show that if $M \in \mathcal{L}(X, X)$ is invertible and similar to $K \in \mathcal{L}(X, X)$, then $K$ is also invertible, and $K^{-1}$ is similar to $M^{-1}$.
   (b) Take $A, B \in \mathcal{L}(X, X)$. Assuming either $A$ or $B$ is invertible, prove $AB$ and $BA$ are similar.