Homework 4 – Advanced Linear Algebra I (MATH 4377/6308) — Fall 2013

Due Thursday September 26 at 11:30AM. Hand in at lecture.

Start early! You may work with others, but the solutions you hand in must be your own. Do not copy others’ solutions! Please show all of the steps necessary to complete the written problem.

1. Read Chapter 2: Duality (pp.13-18) in Lax

2. Given a nonzero vector \( v \in X \), show that there is a linear function \( l \) such that \( l(v) \neq 0 \).
   The elements of \( v \) can be represented as \( (c_1, \ldots, c_n) \). Since \( v \neq 0 \), then there are some \( c_j \neq 0 \). Then create the linear function \( l(v) = a_1c_1 + \cdots + a_nc_n \) by setting \( a_j = c_j \). Then \( l(v) = \sum_{j=1}^{n} c_j^2 > 0 \), since there are some \( c_j \neq 0 \).

3. Let \( Y \) be a subspace of \( X \). Prove that the annihilator \( Y^\perp \) is a subspace of \( X' \).
   By definition, any \( l : X \to \mathbb{R} \) is in \( X' \). Since \( \forall l \in Y^\perp, l : X \to 0 \in \mathbb{R} \), then \( Y^\perp \) is certainly a subset of \( X' \).
   To prove \( Y^\perp \) is a subspace, note that if \( l, m \in Y^\perp \) then \( l(x) = 0 \) and \( m(x) = 0 \ \forall x \in X \). Since \( l, m \) are linear, then \( 0 = l(x) + m(x) = (l + m)(x) \), so \( l + m \in Y^\perp \), so \( Y^\perp \) is closed under addition. Note also \( \forall a \in \mathbb{R} \) and \( \forall l \in Y^\perp, al(x) = a \cdot 0 = 0 \), so \( al \in Y^\perp \), so \( Y^\perp \) is closed under scalar multiplication. So \( Y^\perp \) is a subspace.

4. (a) Let \( V \) be the subspace of \( \mathbb{R}^4 \) spanned by \( v_1 = (-2, 0, 1, 3) \) and \( (1, 1, 1, 1) \). Which linear functions \( l(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \) are in the annihilator of \( V \)?
   These linear functions can be expressed as the null vectors of the matrix
   \[
   \begin{pmatrix}
   -2 & 0 & 1 & 3 \\
   1 & 1 & 1 & 1 
   \end{pmatrix}
   \]
   which can be found by solving the augmented matrix system
   \[
   \begin{pmatrix}
   -2 & 0 & 1 & 3 & | & 0 \\
   1 & 1 & 1 & 1 & | & 0 
   \end{pmatrix} \to \begin{pmatrix}
   -2 & 0 & 1 & 3 & | & 0 \\
   3 & 1 & 0 & 2 & | & 0 
   \end{pmatrix}
   \]
   Setting \( c_1 = 1 \) and \( c_2 = 0 \), we have \( c_3 = -5/2 \) and \( c_4 = 3/2 \). Setting \( c_1 = 0 \) and \( c_2 = 1 \), we have \( c_3 = -3/2 \) and \( c_4 = 1/2 \). Thus
   \[
   l_1(x) = x_1 - 5x_3/2 + 3x_4/2 \\
   l_2(x) = x_2 - 3x_3/2 + x_4/2 
   \]
   are in the annihilator of \( V \).
   (b) Show that \( v_1 \) and \( v_2 \) are in \( V^\perp \), the annihilator of \( V \).
   \[
   \langle v_1, l_1 \rangle = -2 - 5/2 + 9/2 = 0; \quad \langle v_1, l_2 \rangle = -3/2 + 3/2 = 0 \\
   \langle v_2, l_1 \rangle = 1 - 5/2 + 3/2 = 0; \quad \langle v_2, l_2 \rangle = 1 - 3/2 + 1/2 = 0
   \]
5. (a) Determine the weights \(m_1, m_2, m_3\) and \(m_4\) so that

\[
\int_{-1}^{1} p(y) \, dy = m_1 p(-a) + m_2 p(-b) + m_3 p(b) + m_4 p(a)
\]

holds for all polynomials of maximum degree 3: \(p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3\).

Plug in the polynomial to yield

\[
\int_{-1}^{1} c_0 + c_1 y + c_2 y^2 + c_3 y^3 \, dy = m_1 (c_0 - c_1 a + c_2 a^2 - c_3 a^3) + m_2 (c_0 - c_1 b + c_2 b^2 - c_3 b^3) + m_3 (c_0 + c_1 b + c_2 b^2 + c_3 b^3) + m_4 (c_0 + c_1 a + c_2 a^2 + c_3 a^3)
\]

Integrating, we have

\[
2c_0 + 2c_2/3 = m_1 (c_0 - c_1 a + c_2 a^2 - c_3 a^3) + m_2 (c_0 - c_1 b + c_2 b^2 - c_3 b^3) + m_3 (c_0 + c_1 b + c_2 b^2 + c_3 b^3) + m_4 (c_0 + c_1 a + c_2 a^2 + c_3 a^3)
\]

Isolating terms according to coefficients \(c_j\), we have

\[
\begin{align*}
c_0 : & \quad m_1 + m_2 + m_3 + m_4 = 2 \\
c_1 : & \quad -a m_1 - b m_2 + b m_3 + a m_4 = 0 \\
c_2 : & \quad a^2 m_1 + b^2 m_2 + b^2 m_3 + a^2 m_4 = 2/3 \\
c_3 : & \quad -a^3 m_1 - b^3 m_2 + b^3 m_3 + a^3 m_4 = 0
\end{align*}
\]

which we can place into augmented matrix form

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 \\
-a & -b & b & a & 0 \\
a^2 & b^2 & b^2 & a^2 & 2/3 \\
a^3 & -b^3 & b^3 & a^3 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 \\
0 & a-b & a+b & 2a & 2a \\
0 & b^2 - a^2 & b^2 - a^2 & 0 & 2/3 - 2a^2 \\
0 & a^3 - b^3 & a^3 + b^3 & 2a^3 & 2a^3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 \\
0 & a-b & a+b & 2a & 2a \\
0 & 0 & 2b^2 + 2ab & 2a(a+b) & 2/3 + 2ab \\
0 & 0 & -2ab(a+b) & -2ab(a+b) & -2ab(a+b)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 \\
0 & a-b & a+b & 2a & 2a \\
0 & 0 & 2b^2 + 2ab & 2a(a+b) & 2/3 + 2ab \\
0 & 0 & 0 & 2a(a^2 - b^2) & 2a/3 - 2ab^2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 \\
0 & a-b & a+b & 2a & 2a \\
0 & 0 & 0 & 2a(a^2 - b^2) & 3a^2 - 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & (1 - 3b^2)/(3(a^2 - b^2)) \\
0 & 1 & 0 & 0 & (3a^2 - 1)/(3(a^2 - b^2)) \\
0 & 0 & 1 & 0 & (3a^2 - 1)/(3(a^2 - b^2)) \\
0 & 0 & 0 & 1 & (1 - 3b^2)/(3(a^2 - b^2))
\end{pmatrix}
\]
so \( m_1 = m_4 = (1 - 3b^2)/(3(a^2 - b^2)) \), \( m_2 = m_3 = (3a^2 - 1)/(3(a^2 - b^2)) \)

(b) For what values of \( a \) and \( b \) are the weights in (a) positive?

(i) Assume \( a > b > 0 \), then the denominator \( 3(a^2 - b^2) > 0 \) for \( m_j \forall j \). Thus, we have the additional requirements \( 3a^2 - 1 > 0 \) so \( a > \sqrt{1/3} \) and \( 1 - 3b^2 > 0 \) so \( b < \sqrt{1/3} \).

(ii) Assume \( b > a > 0 \), then the denominator \( 3(a^2 - b^2) < 0 \) for \( m_j \forall j \). Thus, we have the additional requirements \( 3a^2 - 1 < 0 \) so \( a < \sqrt{1/3} \) and \( 1 - 3b^2 < 0 \) so \( b > \sqrt{1/3} \).

Note to grader: Just having a single one of these cases is enough for full credit, as long as the student states their assumption. Also, \( a < b < 0 \) cases are OK too.

6. Show that when \( T, R \in \mathcal{L}(X, Y) \) and \( S \in \mathcal{L}(Y, Z) \), then

(a) \( (ST)' = T'S' \);
(b) \( (T + R)' = T' + R' \);
(c) \( (T^{-1})' = (T')^{-1} \).

(a) We will make use of the definition of the transpose \( T' \) with the bilinear form \( (T'l, x) = (l, Tx) \).

This means \( (l, STx) = ((ST)'l, x) \) by definition, but also \( (l, STx) = (S'l, Tx) = (T'S'l, x) \), so \((ST)'l, x) = (T'S'l, x) \forall x \in X \) and \( \forall l \in X' \), so \((ST)' = T'S' \).

(b) \((l, (T + R)x) = ((T + R)'l, x) \) and also \((l, (T + R)x) = (l, Tx) + (l, Rx) = (T'l, x) + (R'l, x) = ((T' + R')l, x) \), so \( (T + R)' = T' + R' \).

(c) \((T^{-1})'l, x) = ((T^{-1}T)'l, x) = (l, x) \forall x \in X \) and \( \forall l \in X' \)

\( (T'(T^{-1})'l, x) = ((T'-(T')^{-1})l, x) = (l, x) \forall x \in X \) and \( \forall l \in X' \)

Therefore, \((T^{-1})' \) satisfies all properties of being an inverse of \( T' \), so \((T^{-1})' = (T')^{-1} \)

7. (a) Show that if \( M \) is invertible and similar to \( K \), then \( K \) is also invertible, and \( K^{-1} \) is similar to \( M^{-1} \).

We have that there is some invertible \( S \) such that \( M = SKS^{-1} \) and an inverse \( M^{-1} \) exists such that \( MM^{-1} = I \). Note \( K = S^{-1}MS \) and \((S^{-1}M^{-1}S)K = (S^{-1}M^{-1}S)S^{-1}MS = S^{-1}M^{-1}MS = S^{-1}1 = S^{-1}S = I \), so \( K \) is invertible and \( K^{-1} = S^{-1}M^{-1}S \), which is similar to \( M^{-1} \).

(b) Take \( A, B \in \mathcal{L}(X, X) \). Show that if either \( A \) or \( B \) is invertible, then \( AB \) and \( BA \) are similar.

If \( A \) is invertible, then \( A^{-1}(AB)A = BA \), so \( AB \) and \( BA \) are similar.

If \( B \) is invertible, then \( B^{-1}(BA)B = AB \), so \( BA \) and \( AB \) are similar.